ABSTRACT

Part II of this paper is a direct continuation of Part I, where we consider the same types of orthorhombic layered media and the same types of pure-mode and converted waves. Like in Part I, the approximations for the slowness-domain kinematical characteristics are obtained by combining power series coefficients in the vicinity of both the normal-incidence ray and an additional wide-angle ray. In Part I, the wide-angle ray was set to be the critical ray (‘critical slowness match’), whereas in Part II we consider a finite long offset associated with a given pre-critical ray (‘pre-critical slowness match’). Unlike the critical slowness match, the approximations in the pre-critical slowness match are valid only within the bounded slowness range; however, the accuracy within the defined range is higher. Moreover, for the pre-critical slowness match, there is no need to distinguish between the high-velocity layer and the other, low-velocity layers. The form of the approximations in both critical and pre-critical slowness matches is the same, where only the wide-angle power series coefficients are different. Comparing the approximated kinematical characteristics with those obtained by exact numerical ray tracing, we demonstrate high accuracy. Furthermore, we show that for all wave types, the accuracy of the pre-critical slowness match is essentially higher than that of the critical slowness match, even for matching slowness values close to the critical slowness. Both approaches can be valuable for implementation, depending on the target offset range and the nature of the subsurface model. The pre-critical slowness match is more accurate for simulating reflection data with conventional offsets. The critical slowness match can be attractive for models with a dominant high-velocity layer, for simulating, for example, refraction events with very long offsets.

Key words: Anisotropy, Modeling.
to be the critical ray (critical slowness match). It was assumed that the model includes a dominant high-velocity layer, and that for nearly critical rays, most of the contributions to the offset and traveltime are due to the propagation in the high-velocity layer. In this work, Part II, the assumption of a dominant high-velocity layer is removed, and instead of the critical ray, a long-offset pre-critical ray is used. We derive the azimuthally varying slowness-domain kinematical characteristics (radial and transverse offset components, intercept time and travelt ime) for compressional and converted waves, where the detailed derivations are described within the appendices. In Appendices A and B, we derive all partial derivatives of the intercept time, needed for the pre-critical slowness match, including the mixed derivatives, and the higher derivatives. In Appendix C, we discuss the limit case of the pre-critical slowness match, and compare it with the critical slowness match.

Using the same synthetic models as in Part I, we demonstrated that the accuracy of the pre-critical slowness match is higher than the accuracy of the critical slowness match within the given bounded slowness range.

Although this two-part paper is devoted to the forward computation of the kinematical characteristics for layered orthorhombic media, we have added a section describing the main role of the kinematical characteristics and the corresponding set of effective parameters within the seismic inversion workflow.

### APPROPRIATIONS FOR THE KINEMATICAL CHARACTERISTICS

The approximation formulae for the critical slowness match presented in Part I and the pre-critical slowness match to be used in this part are identical. The azimuthally dependent coefficients $A, B, C, D, E$ are different and will be derived in this work. The expressions for the slowness-domain kinematical characteristics presented in Part I are given as:

- For the radial offset,

\[
\hat{b}_R (\bar{\theta}_b, \psi_{slw}) = \frac{A (\psi_{slw}) \hat{b}_\psi + B (\psi_{slw}) \hat{b}_\psi^2 + C (\psi_{slw}) \hat{b}_\psi^3}{\bar{q}} + D (\psi_{slw}) \hat{b}_\psi + E (\psi_{slw}) \hat{b}_\psi^2. \tag{1}
\]

- For the intercept time,

\[
\tau (\bar{\theta}_b, \psi_{slw}) - t_c (\psi_{slw}) = -A (\psi_{slw}) (1 - \bar{q}) - \frac{B (\psi_{slw}) (1 - \bar{q})^2 (2 + \bar{q})}{3} - \frac{C (\psi_{slw}) (1 - \bar{q})^3 (8 + 9 \bar{q} + 3 \bar{q}^2)}{15} - \frac{D (\psi_{slw}) \hat{b}_\psi^2}{2} - \frac{E (\psi_{slw}) \hat{b}_\psi^3}{4}. \tag{2}
\]

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**Table 1** Relative errors of the approximations for the single-layer model

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<th>$\hat{b}_{h,f}$</th>
<th>Wave Type</th>
<th>$b_{ave}/z$</th>
<th>Radial Offset</th>
<th>Transverse</th>
<th>Intercept Time</th>
<th>Traveltime</th>
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Table 2 Relative errors of the approximations for the multi-layer model

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<th>$b_{ave}/z$</th>
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<th>Intercept Time</th>
<th>Traveltime</th>
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<td>$1.644 \times 10^{-4}$</td>
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</tr>
</tbody>
</table>

- For the traveltime,

\[ t(p_{h},\psi_{slw}) - t_{n} = \frac{A(\psi_{slw})(1 - \tilde{q})}{p_{c}(\psi_{slw})} \]
\[ + \frac{B(\psi_{slw})(1 - \tilde{q})^{2}(3 + 4\tilde{q} + 2\tilde{q}^{2})}{3\tilde{q}} \]
\[ + \frac{C(\psi_{slw})(1 - \tilde{q})^{2}(3 + 2\tilde{q})(5 + 9\tilde{q} + 6\tilde{q}^{2})}{15\tilde{q}} \]
\[ + \frac{D(\psi_{slw})}{2} + 3E(\psi_{slw}) \frac{p_{h}^{2}}{4}. \]  

(3)

- For the transverse offset,

\[ b_{T}(p_{h},\psi_{slw}) = - \frac{t(\psi_{slw}, p_{b}) - t_{0}}{p_{b}^{2}p_{c}(\psi_{slw})} + A(\psi_{slw})(1 - \tilde{q}) \]
\[ + B(\psi_{slw}) \frac{(1 - \tilde{q})^{2}(2 + \tilde{q})}{3p_{b}} + C(\psi_{slw}) \frac{(1 - \tilde{q})^{2}(8 + 9\tilde{q} + 3\tilde{q}^{2})}{15p_{b}} \]
\[ + D(\psi_{slw}) \frac{p_{b}}{2} + E(\psi_{slw}) \frac{p_{b}^{2}}{4}. \]  

(4)

where $A'$, $B'$, $C'$, $D'$ are derivatives of coefficients $A$, $B$, $C$, $D$ with respect to the slowness azimuth $\psi_{slw}$, $p_{b}$ and $p_{c}$ are the non-normalized and normalized horizontal slowness, respectively, $\tilde{q}$ is the normalized complementary slowness,

\[ \tilde{p}_{b}(\psi_{slw}) = \frac{p_{b}}{p_{c}(\psi_{slw})}, \quad \tilde{q}(\psi_{slw}) = \sqrt{1 - \tilde{p}_{b}^{2}(\psi_{slw})}, \]  

(5)

and $p_{c}(\psi_{slw})$ is the critical slowness at a given slowness azimuth $\psi_{slw}$.

**COEFFICIENTS FOR THE PRE-CRITICAL SLOWNESS MATCH**

The resolving set of five equations needed in order to find the five azimuthally dependent coefficients $A$, $B$, $C$, $D$, $E$ for the pre-critical slowness match differs from that for the critical slowness match. The first two equations are still related to the normal-incidence ray, whose second- and fourth-order normal moveout (NMO) velocities, $V_{2}(\psi_{slw})$ and $V_{4}(\psi_{slw})$, can be
Slowness-domain kinematical characteristics for layered orthorhombic media

Considered the total two-way intercept time $\tau$ and its first and second derivatives $\tau'$, $\tau''$ with respect to the normalized horizontal slowness $p_h$, the three additional conditions are

$$ \tau (p_h, \psi_{slw}) ; $$

(7a)

$$ \tau' (p_h, \psi_{slw}) = \frac{\partial \tau (p_h, \psi_{slw})}{\partial p_h} = \frac{\partial \tau (p_h, \psi_{slw})}{\partial p_h} \frac{dp_h}{dp_h} ; $$

(7b)

$$ \tau'' (p_h, \psi_{slw}) = \frac{\partial^2 \tau (p_h, \psi_{slw})}{\partial p_h^2} = \frac{\partial^2 \tau (p_h, \psi_{slw})}{\partial p_h^2} p_h^2 (\psi_{slw}) . $$

(7c)

Thus, for a given matching horizontal slowness magnitude $p_h$ and azimuth $\psi_{slw}$, the values of the intercept time and its derivatives with respect to the normalized horizontal slowness are specified.
The first derivative of the intercept time with respect to the horizontal slowness $p_h$ is given by
\[
\frac{\partial \tau (p_h, \psi_{slw})}{\partial p_h} = \frac{1}{p_c (\psi_{slw})} \frac{\partial \tau (\psi_{slw}, p_h)}{\partial p_h} = \frac{A (\psi_{slw})}{\bar{q}} (1 + 4 \bar{q}^2) \left[ A (\psi_{slw}) + B (\psi_{slw}) \bar{p}_h (1 + 2 \bar{q}^2) + C (\psi_{slw}) \bar{p}_h^5 (1 + 4 \bar{q}^2) \right] - D (\psi_{slw}) \bar{q}^3 - 3 E (\psi_{slw}) \bar{p}_h^2,
\]
which is an obvious formula, because
\[
\frac{\partial \tau (p_h, \psi_{slw})}{\partial p_h} = - b_h (p_h, \psi_{slw}).
\]

Next we compute the second derivative of the intercept time,
\[
\frac{1}{p_c (\psi_{slw})} \frac{\partial^2 \tau (p_h, \psi_{slw})}{\partial \bar{p}_h^2} = \frac{A (\psi_{slw})}{\bar{q}} (1 + 4 \bar{q}^2) \left[ A (\psi_{slw}) + B (\psi_{slw}) \bar{p}_h (1 + 2 \bar{q}^2) + C (\psi_{slw}) \bar{p}_h^5 (1 + 4 \bar{q}^2) \right] - D (\psi_{slw}) \bar{q}^3 - 3 E (\psi_{slw}) \bar{p}_h^2,
\]
(10)

We introduce parameters $\bar{p}_{h, f}$ and $\bar{p}_{h, f}$ for the normalized and non-normalized pre-critical matching horizontal slowness, respectively,
\[
\bar{p}_{h, f} = \bar{p}_{h, f} \bar{p}_c (\psi_{slw}) \bar{q}_{f} = \sqrt{1 - \bar{p}_{h, f}^2},
\]
(11)
where $\bar{q}_{f}$ is the normalized complementary matching slowness. Combining equations (2), (6), (8) and (10) and introducing the matching horizontal slowness instead of an arbitrary horizontal slowness, $p_h = p_{h, f}$, we obtain the resolving linear set. The five equations can be arranged in the following matrix form:
\[
M (\bar{p}_{h, f}) \cdot \begin{bmatrix} A (\psi_{slw}) \n B (\psi_{slw}) \n C (\psi_{slw}) \n D (\psi_{slw}) \n E (\psi_{slw}) \end{bmatrix} = J (\psi_{slw}, \bar{p}_{h, f}).
\]
(12)
The 5 × 5 matrix on the left-hand side of equation (12) reads

\[
M(p_{h,f}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
1 - \bar{q}_f & \frac{\bar{q}_f}{q_f} (1 - \bar{q}_f)^2 & \frac{\bar{q}_f}{q_f} (1 - \bar{q}_f)^2 & \frac{\bar{q}_f}{q_f} \frac{\bar{q}_f}{q_f} & \frac{\bar{q}_f}{q_f} \frac{\bar{q}_f}{q_f} \\
\frac{\bar{q}_f}{q_f} & \frac{\bar{q}_f}{q_f} & \frac{\bar{q}_f}{q_f} & \frac{\bar{q}_f}{q_f} & \frac{\bar{q}_f}{q_f} \\
\frac{1}{q_f} & \frac{1}{q_f} & \frac{1}{q_f} & \frac{1}{q_f} & \frac{1}{q_f} \\
\end{bmatrix}, \quad (13)
\]

and the right-hand side vector reads

\[
J(p_{h,f}, \psi_{slw}) = \begin{bmatrix}
V_{2}^2(\psi_{slw}) p_{h}(\psi_{slw}) t_0 \\
V_{2}^2(\psi_{slw}) p_{h}(\psi_{slw}) t_0 \\
\frac{\partial}{\partial p_h}(p_{h}(\psi_{slw})) \\
\frac{\partial}{\partial p_h}(p_{h}(\psi_{slw})) \\
\frac{\partial^2}{\partial p_h^2}(p_{h}(\psi_{slw}))
\end{bmatrix} . \quad (14)
\]

The horizontal slowness \( p_h \) in equation (14) corresponds to the pre-critical matching horizontal slowness \( p_{h,f} \). Solving the linear set of five equations for the given fixed magnitude and azimuth of the horizontal slowness (pre-critical slowness match), we find the unknown azimuthally dependent coefficients \( A', B', C', D', E' \). We note that only the right-hand side vector \( J \) depends on the slowness azimuth. Its components have units of distance. The matrix components are independent of the azimuth. These components are normalized (unitless). The unknown variables also have units of distance. Once the approximation coefficients are obtained, the horizontal slowness in the approximation formulae (1)–(3) can accept any value, not to exceed the matching limit, \( \bar{p}_h \leq \bar{p}_{h,f} \). For the normalized horizontal slowness \( \bar{p}_h \) above the matching value, the errors of the approximations increase rapidly and catastrophically.

\[\text{DERIVATIVES OF THE COEFFICIENTS FOR PRE-CRITICAL SLOWNESS MATCH}\]

The five coefficients of the approximations for the radial offset, intercept time and traveltine depend on the slowness azimuth. However, in order to obtain the transverse offset component, we also need the azimuthal derivatives of these coefficients, \( A', B', C', D', E' \). First, azimuthal derivatives can...
be computed from the equations for the second- and fourth-order normal moveout (NMO) velocities,

\[ A' (\psi_{slw}) + D' (\psi_{slw}) = \left[ V_c^2 (\psi_{slw}) \right] p_c (\psi_{slw}) t_o + V_c^2 (\psi_{slw}) p' c (\psi_{slw}) t_o, \]

\[ A' (\psi_{slw}) + 2B' (\psi_{slw}) + 2E' (\psi_{slw}) = \left[ V_c^4 (\psi_{slw}) \right] p_c^3 (\psi_{slw}) t_o + 3 V_c^4 (\psi_{slw}) p_c^2 (\psi_{slw}) p' c (\psi_{slw}) t_o. \]

(15)

Next, we differentiate equations (2), (8) and (10) (for the intercept time \( \tau \) and its derivatives \( \partial \tau / \partial p, \partial^2 \tau / \partial p^2 \)) with respect to the slowness azimuth \( \psi_{slw} \), keeping the normalized horizontal slowness \( \bar{p}_h \) constant. Constant \( \bar{p}_h \) also means constant normalized complementary slowness \( \bar{q} \), because \( \bar{p}_h^2 + \bar{q}^2 = 1 \).

The azimuthal derivative of the intercept time reads

\[ \frac{1}{p_c (\psi_{slw})} \frac{\partial \tau (p_h, \psi_{slw})}{\partial \psi_{slw}} \bigg|_{p_h = \text{const}} = \tau (p_h, \psi_{slw}) - t_o p' c (\psi_{slw}) \]

\[ - A' (\psi_{slw}) (1 - \bar{q}) - \frac{B' (\psi_{slw}) (1 - \bar{q})^2 (2 + \bar{q})}{3} - C' (\psi_{slw}) (1 - \bar{q})^3 (8 + 9\bar{q} + 3\bar{q}^3) \]

\[ - \frac{D' (\psi_{slw}) \bar{p}_h^2}{15} \]

\[ - \frac{E' (\psi_{slw}) \bar{p}_h^4}{4}. \]

(16)

The mixed second derivative reads

\[ \frac{\partial^2 \tau (p_h, \psi_{slw})}{\partial p_h \partial \psi_{slw}} \bigg|_{p_h = \text{const}} = \frac{\partial \tau (p_h, \psi_{slw})}{\partial p_h} p' c (\psi_{slw}) + \frac{A' (\psi_{slw}) \bar{p}_h + B' (\psi_{slw}) \bar{p}_h^3 + C' (\psi_{slw}) \bar{p}_h^5 \bar{q}}{\bar{q}} \]

\[ - D' (\psi_{slw}) \bar{p}_h - E' (\psi_{slw}) \bar{p}_h^3. \]

(17)

The mixed third derivative reads

\[ \frac{\partial^3 \tau (p_h, \psi_{slw})}{\partial p_h^2 \partial \psi_{slw}} \bigg|_{p_h = \text{const}} = \frac{\partial^2 \tau (p_h, \psi_{slw})}{\partial p_h^2} p' c (\psi_{slw}) + \frac{A' (\psi_{slw}) \bar{p}_h + B' (\psi_{slw}) \bar{p}_h^3 + C' (\psi_{slw}) \bar{p}_h^5 (1 + 4\bar{q}^2)}{\bar{q}^3} \]

\[ - D' (\psi_{slw}) - 3 E' (\psi_{slw}) \bar{p}_h^2. \]

(18)
where the radial offset, transverse offset, intercept time and traveltime.

Combining equations (15)–(18), we obtain the linear set,

$$
\begin{align*}
M \left( \bar{p}_h, \psi_{slw} \right) & = J \left( \psi_{slw}, p_h, f \right), \\
\end{align*}
$$

where the $5 \times 5$ matrix $M \left( \bar{p}_h, \psi_{slw} \right)$ is the same as in equation (12), and its components are listed in equation (13). The right-hand side vector is different,

$$
J \left( p_h, f, \psi_{slw} \right) = \begin{bmatrix}
\tau \left( p_h, \psi_{slw} \right) - \bar{\tau}_o, \\
\frac{\partial \tau \left( p_h, \psi_{slw} \right)}{\partial p_h} \bar{p}_h, \\
\frac{\partial \tau \left( p_h, \psi_{slw} \right)}{\partial \psi_{slw}} \frac{\partial \psi_{slw}}{\partial p_h} \bar{p}_h, \\
\frac{\partial \tau \left( p_h, \psi_{slw} \right)}{\partial \psi_{slw}} \frac{\partial \psi_{slw}}{\partial p_h} \bar{p}_h, \\
\frac{\partial \tau \left( p_h, \psi_{slw} \right)}{\partial \psi_{slw}} \frac{\partial \psi_{slw}}{\partial p_h} \bar{p}_h
\end{bmatrix}
$$

The normalized horizontal slowness in equation (20) corresponds to the pre-critical matching value, $p_h = p_h, f$. Once the approximation coefficients are obtained, the horizontal slowness in the approximation formula (4) can accept any value within the range of $p_h \leq p_h, f$. We emphasize that when computing the azimuthal derivatives, keeping the normalized horizontal slowness $p_h$ constant is not the same as keeping the horizontal slowness $p_h$ constant, because the normalizing factor – the critical slowness $p_h, f \left( \psi_{slw} \right)$ – depends on the slowness azimuth. It follows from equation (20) that three such derivatives are needed.
Using the relationships of Appendix A to establish the values for equation (21). Note that the notation for the constant $p_h$ is related to the derivative with respect to the slowness azimuth only. The relationships between the azimuthal derivatives keeping $p_h$ constant and those keeping $p_h$ constant are given in Appendix A. Thus, we first compute regular partial derivatives of the intercept time (either with respect to the horizontal slowness magnitude, keeping the slowness azimuth constant, or vice versa). Next, we apply the relationships of Appendix A to establish the values for equation (21).

We invert matrix $M(p_{h,i})$ in equation (13), and then combining equations (12) and (19), we obtain

$$
\frac{\partial \tau (\psi_{d,w}, p_h)}{\partial \psi_{d,w}} \bigg|_{p_h = \text{const}} \quad \frac{\partial^2 \tau (\psi_{d,w}, p_h)}{\partial \psi_h \partial \psi_{d,w}} \bigg|_{p_h = \text{const}} \quad \frac{\partial^3 \tau (\psi_{d,w}, p_h)}{\partial \psi_h^2 \partial \psi_{d,w}} \bigg|_{p_h = \text{const}}.
$$

Note that the notation for the constant $p_h$ is related to the derivative with respect to the slowness azimuth only. The relationships between the azimuthal derivatives keeping $p_h$ constant and those keeping $p_h$ constant are given in Appendix A. Thus, we first compute regular partial derivatives of the intercept time (either with respect to the horizontal slowness magnitude, keeping the slowness azimuth constant, or vice versa). Next, we apply the relationships of Appendix A to establish the values for equation (21).

We invert matrix $M(p_{h,i})$ in equation (13), and then combining equations (12) and (19), we obtain

$$
\begin{bmatrix}
A(p_{h,f}, \psi_{d,w}) \\
B(p_{h,f}, \psi_{d,w}) \\
C(p_{h,f}, \psi_{d,w}) \\
D(p_{h,f}, \psi_{d,w}) \\
E(p_{h,f}, \psi_{d,w})
\end{bmatrix}
\begin{bmatrix}
A'(p_{h,f}, \psi_{d,w}) \\
B'(p_{h,f}, \psi_{d,w}) \\
C'(p_{h,f}, \psi_{d,w}) \\
D'(p_{h,f}, \psi_{d,w}) \\
E'(p_{h,f}, \psi_{d,w})
\end{bmatrix}
= M^{-1}(p_{h,f})
\begin{bmatrix}
J_1(p_{h,f}, \psi_{d,w}) \\
J_2(p_{h,f}, \psi_{d,w}) \\
J_3(p_{h,f}, \psi_{d,w}) \\
J_4(p_{h,f}, \psi_{d,w}) \\
J_5(p_{h,f}, \psi_{d,w})
\end{bmatrix}.
$$

where $J_i(p_{h,f}, \psi_{d,w})$ and $J'_i(p_{h,f}, \psi_{d,w}), i = 1, 2, \ldots, 5$, are components of the right-hand side vectors in equation sets (12) and (19), respectively (listed in equations (14) and (20)).

**Derivatives of the Intercept Time**

For the constant properties of each layer, the two-way intercept time can be presented as a sum of intercept times for the individual layers,

$$
\tau = \sum_{i=1}^{n} \Delta \tau_i, \quad \Delta \tau_i = \Delta z_i (p_{ci} + p_{ci}^{'})\quad \text{.}
$$

where $i$ is the layer index, $\Delta \tau_i$ is the two-way intercept time of layer $i$, $\Delta z_i$ is the layer thickness and $n$ is the number of layers in the model. Parameters $p_{ci}^0$ and $p_{ci}'$ represent the vertical slowness of the $i$th layer for the incident and reflected waves (both are positive values). They depend on the wave
mode, and, therefore, they are identical for pure-mode waves and different for converted waves. Thus, to obtain the derivatives of the intercept time, we just compute the derivatives of the slowness surface (the derivatives of the vertical slowness with respect to the magnitude and azimuth of the horizontal slowness), and stack these derivatives with the corresponding weights; each weight is the layer thickness $\Delta z_i$. The derivatives of the slowness surface, for all wave types propagating within the layered orthorhombic media, are obtained analytically by implicit differentiation of the Christoffel polynomial equation, as described in Appendix B.

SYNTHETIC EXAMPLES

To test the accuracy of the pre-critical slowness match for the kinematical characteristics (the radial and transverse offset, intercept time and traveltime), we compute the relative errors of the approximations comparing them with the values computed by numerical ray tracing. We consider the same homogeneous and layered orthorhombic models as used in Part I, studying compressional waves and two types of converted waves, P–S1 and P–S2. The elastic and geometric properties of the layers for the two models are listed in Tables 1 and 2 in Part I. The error plots and maximum error tables demonstrate higher accuracy of the pre-critical slowness match that decreases as the matching slowness increases.

Accuracy of the homogeneous model

Figures 1, 2 and 3 demonstrate the accuracy of the approximations for compressional waves, for the normalized matching horizontal slowness $p_h = 0.97$, respectively. Figures 4, 5 and 6 demonstrate the accuracy for P–S1 converted waves for the same matching horizontal slowness, and Figs 7, 8 and 9 for P–S2 converted waves. The relative errors are plotted for all slowness azimuths and for all magnitudes of the horizontal slowness, up to the matching values.

Accuracy of the layered model

Figures 10, 11 and 12 demonstrate the accuracy of the approximations for compressional waves, for the normalized
Figure 8 Accuracy of the approximations for P–S2 converted wave, single-layer model, normalized matching horizontal slowness $\bar{p}_h = 0.95$: (a) radial offset, (b) transverse offset, (c) intercept time and (d) traveltime.

The matching horizontal slowness values $\bar{p}_{h,f} = [0.999, 0.998, 0.995]$, respectively. Figures 13, 14 and 15 demonstrate the accuracy for P–S1 converted waves for the same matching horizontal slowness, and Figs 16, 17 and 18 for P–S2 converted waves. For the pre-critical slowness match, the same tendency holds as for the critical slowness match. The intercept time approximation is more accurate than the approximations for the radial offset and the traveltime, because the integration operator smoothens off the errors/inaccuracies of the function under the integral operator.

Tables 1 and 2 summarize the maximum relative errors for all magnitudes and azimuths of the horizontal slowness for the homogeneous and layered orthorhombic models, respectively. The first column in these tables shows the matching pre-critical slowness. Critical slowness match is considered a particular case. The second column is the wave type.

The normalized matching horizontal slowness $\bar{p}_{h,f}$ is a suitable computational parameter; however, the offset-to-depth ratio is more informative. Given a fixed normalized or non-normalized horizontal slowness ($p_h$ or $\bar{p}_h$), the offset-to-depth ratio in an orthorhombic model becomes azimuthally dependent. However, the exact value for each azimuth is not that essential, and therefore, we computed the average value for all slowness azimuths (column 3). The incidence and reflection angles are also azimuthally dependent for constant $\bar{p}_h$.

Note that the normalized slowness matching values (0.999, 0.998 and 0.995) applied for the layered model were intentionally chosen much closer to the critical slowness than the matching values that we used for the homogeneous model (0.97, 0.95 and 0.92). For a multi-layer model, when $\bar{p}_h$ approaches 1, the offset-to-depth ratio increases with a smaller rate than for a single layer. The reason is the low-velocity layers, whose contribution to the offset (and traveltime) remains finite as the normalized slowness approaches 1. Columns 4–7 show the maximum relative errors for the four kinematical characteristics. The greatest errors are indicated in bold font and correspond to the critical slowness match.

The accuracy of the approximations for the pre-critical slowness match is much higher than that for the critical slowness match, even in the cases when the matching slowness differs from the critical value by a couple of per cent (single-layer model) or even a small fraction of a per cent (multi-layer model).
DISCUSSION: THE ROLE OF KINEMATIC CHARACTERISTICS IN SEISMIC INVERSION

Seismic modelling and imaging and their use in the inversion of subsurface anisotropic elastic model parameters in realistic subsurface geological areas are mainly based on accurate, time-consuming, wave-equation or ray-tracing methods. These methods require detailed information about the subsurface elastic parameters on relatively fine grids with a resolution size in the order of the wavelength of the prevailing waves. The general workflow for obtaining these models consists of two main stages: (a) building an initial (normally isotropic), long-wavelength background model by directly analysing the time domain prestack seismic data and (b) an iterative process for updating and refining the intermediate velocity models. The two main commonly used updating approaches are wave-equation-based tomography, or what is usually called full-waveform inversion method (e.g. Tarantola 1984; Tarantola et al. 1988; Symes 2008 and many others), and image-domain ray-based tomography which is a class of solutions within the so-called migration velocity analysis (MVA) (e.g. Stork 1992; Kosloff et al. 1996; Alkhalifah, Fomel and Biondi 1997; Alkhalifah 2003 and many others). Each iteration in the MVA approach involves the following steps: performing seismic migration and generating common image gathers (CIGs), preferably with rich azimuths and long offsets/wide-angles, analysing the residual moveouts (RMOs) at reflection points along the CIG traces, and finally using the RMOs, as the sources of information for traveltime errors along reflected rays, for estimating the updated model parameters. The updated solution can be performed by (preferably) a global ray-based tomographic approach or by a local approach based on computation and analysis of global effective parameters (e.g. Koren and Ravve 2014). Advanced methods normally combine these two approaches, where the estimated global (multi-layer) effective parameters are used for obtaining local (layer) effective parameters, which are then used to provide first estimates and physical constraints for the inverted interval elastic parameters. In order to describe the role of the anisotropic kinematic characteristics and the corresponding effective parameters in this combined...
Figure 10 Accuracy of the approximations for compressional wave, multi-layer model, normalized matching horizontal slowness $\bar{p}_h = 0.999$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 11 Accuracy of the approximations for compressional wave, multi-layer model, normalized matching horizontal slowness \( \bar{p}_h = 0.998 \): (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.

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Figure 12 Accuracy of the approximations for compressional wave, multi-layer model, normalized matching horizontal slowness $\bar{\rho}_h = 0.995$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 13  Accuracy of the approximations for P–S1 converted wave, multi-layer model, normalized matching horizontal slowness $\bar{p}_h = 0.999$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.

Figure 14  Accuracy of the approximations for P–S1 converted wave, multi-layer model, normalized matching horizontal slowness $\bar{p}_h = 0.998$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 15 Accuracy of the approximations for P–S1 converted wave, multi-layer model, normalized matching horizontal slowness $\bar{\mu} = 0.995$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 16 Accuracy of the approximations for P–S2 converted wave, multi-layer model, normalized matching horizontal slowness $\bar{p}_h = 0.999$: 
(a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 17 Accuracy of the approximations for P–S2 converted wave, multi-layer model, normalized matching horizontal slowness $p_h = 0.998$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
Figure 18  Accuracy of the approximations for P–S2 converted wave, multi-layer model, normalized matching horizontal slowness $\bar{p}_h=0.995$: (a) radial offset in the first azimuthal quadrant, (b) radial offset in the second quadrant, (c) transverse offset in the first quadrant, (d) transverse offset in the second quadrant, (e) intercept time in the first quadrant, (f) intercept time in the second quadrant, (g) traveltime in the first quadrant and (h) traveltime in the second quadrant.
work, let us consider the following example: Assume a mature iterative stage, where a background (roughly estimated) layered orthorhombic model has already been derived, mainly from rich-azimuth and long-offset seismic data and some additional a priori information. Running another MVA-driven iteration with the background orthorhombic model, we first perform a migration to generate full-azimuth reflection angle gathers (e.g. Koren and Ravve 2011; Ravve and Koren 2011). Looking at the resulted image gathers, we still see periodic azimuthal residual moveouts, especially along wide-angle image traces, which indicate that the accuracy of the background orthorhombic velocity field used in the migration is not sufficient. Another updated iteration is required. In the context of using the azimuthally varying effective velocity parameters (used to approximate the kinematical characteristics), the following steps are required:

1. Forward computation of the azimuthally varying effective parameters, or the five coefficients, $A_{bg}^\psi(\psi_{slw})$, $\ldots$, $E_{bg}^\psi(\psi_{slw})$ plus the critical slowness $p_{bg}^\psi(\psi_{slw})$, described in this two-part paper, for the background (migration-based) orthorhombic model. The superscript ‘bg’ indicates that the computation is performed on the ‘background’ model.

2. Analysing the RMOs: This stage requires first obtaining an explicit RMO formula for each normal-incidence (vertical) time and a given slowness azimuth. The explicit formula for the vertical time-scaled RMO, $\Delta t(p_b, \psi_{slw}; \tau_o; p_c, \Delta A, \Delta B, \Delta C, \Delta D, \Delta E)$, is beyond the scope of this study. Although not essential, for simplicity, during the derivation of the RMO formula we assume that the critical slowness $p_c(\psi_{slw})$ does not change and it corresponds to the background model. $\Delta A, \ldots, \Delta E$ are the small perturbations of the effective parameters, $A_{up}^\psi = A_{bg}^\psi + \Delta A$ (and so forth all other parameters), where the superscript ‘up’ indicates ‘updated’ parameters. The explicit RMO formula is derived by setting to zero the full differential of the traveltime expression (preservation of the invariant traveltime of the seismic events) with respect to the normal-incidence (vertical) traveltime $\tau_o$, and all other, azimuthally dependent, effective parameters, $dt(p_b, \psi_{slw}; \tau_o; p_c, A, B, C, D, E) = 0$.

3. The updated parameters, $A_{up}^\psi(\psi_{slw})$, $\ldots$, $E_{up}^\psi(\psi_{slw})$, make it possible to obtain the azimuthally dependent effective velocity parameters, for example, in the case of the ‘critical slowness match’, $V_{up}^2(\psi_{slw})$, $V_{up}^4(\psi_{slw})$, $M_{up}^{\sigma}(\psi_{slw})$, $M_{up}^{\phi}(\psi_{slw})$ and $t_{up}^L(\psi_{slw})$.

4. A constrained generalized Dix-type approach can then be applied to obtain the local effective parameters between any two successive depth (vertical time) values. The azimuthally dependent local second- and fourth-order normal moveout (NMO) velocities $v_u^2(\psi_{slw})$ and $v_u^4(\psi_{slw})$ can then be inverted to provide the high and the low local NMO velocities at each layer, and the local azimuth of the high NMO velocity. A further constrained inversion can be applied to obtain the actual interval orthorhombic parameters and the azimuth of the local vertical symmetry planes.

We believe that one should not underestimate the importance of the derived global and local effective parameters that provide explicit approximation formulae for the kinematical characteristics. Even within the frame of waveform- or ray-based inversions, the effective parameters provide valuable information (e.g. Zhang and Alkhalifah 2017). The seismic method, with all the available data, still cannot uniquely determine the complexity of the subsurface model, especially in areas characterized by considerable heterogeneity and anisotropy effects. The average nature of the effective parameters plays an important role in setting the framework of the model to be derived and the corresponding physical constraints to be applied.

CONCLUSIONS

Considering compressional and converted waves, we propose accurate, azimuthally dependent slowness-domain approximations for the kinematical characteristics: the radial and transverse offset components, intercept time and travel-time for horizontally layered orthorhombic models. The approximations are valid up to a given wide angle range. The approximations are based on merging the series expansion coefficients of the intercept time for two (virtual) rays: the normal-incidence ray and an additional azimuthally dependent wide-angle ray. In Part I, the wide-angle ray was considered the azimuthally varying critical ray, where we showed the effectiveness of this method in cases where the subsurface models include dominant high-velocity layers. In Part II, the requirement for a dominant high-velocity layer is removed, and the azimuthally varying wide-angle ray is associated with a finite long offset related to a given pre-critical slowness. The general form of the approximated formulae is the same; however, the coefficients are different. Within the given pre-critical slowness range, the accuracy of the pre-critical slowness match presented in Part II is higher than the accuracy obtained by the critical slowness match presented in Part I.

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### NOTATIONS

**Offset vector**
- \( h \): horizontal offset vector
- \( b_1, b_2 \): Cartesian offset components
- \( b_R \): radial offset component
- \( \Delta h_{R,H} \): contribution of the high-velocity layer into the radial offset
- \( b_{R,L} \): contribution of the low-velocity layers into the radial offset
- \( \Delta h_{T,H} \): contribution of the high-velocity layer into the transverse offset
- \( b_{T,L} \): contribution of the low-velocity layers into the transverse offset
- \( h \): offset magnitude
- \( \psi_{off} \): offset azimuth

**Slowness vector**
- \( p \): slowness vector
- \( p_c(\psi_{slw}) \): azimuthally dependent critical slowness
- \( m_2, m_6, m_3 \): auxiliary coefficients needed to compute Cartesian frame
- \( p_1, p_2 \): horizontal slowness components
- \( p_3 \equiv \psi \): depth-dependent vertical slowness component (slowness surface)
- \( p^e \): depth-dependent slowness surface of the incidence ray
- \( p^e \): depth-dependent slowness surface of the reflection ray
- \( p_{r,H} \): slowness surface of the high-velocity layer
- \( L_h, L_{ab}, L_{abab}, L_{g}, L_{g-H}, L_{abab} \): auxiliary parameters needed to compute derivatives of the slowness surface
- \( p_h \): horizontal slowness (magnitude)
- \( p_h,f \): pre-critical matching horizontal slowness
- \( \bar{p}_h \): normalized horizontal slowness
- \( \bar{p}_h,f \): normalized pre-critical matching horizontal slowness
- \( q \): complementary slowness
- \( \bar{q} \): normalized complementary slowness
- \( \bar{q}_f \): normalized complementary pre-critical matching slowness
- \( \psi_{slw} \): global slowness azimuth
- \( \Delta \psi_{slw} \): local slowness azimuth, for a specific layer

**Traveltime (two-way if not specified otherwise)**
- \( t \): reflection traveltime
- \( t_o \): vertical (normal-incidence) time
- \( \Delta t_H \): contribution of the high-velocity layer into the reflection traveltime
- \( t_L \): contribution of the low-velocity layers into the reflection traveltime

**Intercept time (two-way if not specified otherwise)**
- \( \tau \): intercept time
- \( \Delta \tau_H \): contribution of the high-velocity layer into the intercept time
- \( \tau_L \): contribution of the low-velocity layers into the intercept time

**Global effective parameters (related to the entire multi-layer model)**
- \( A(\psi_{slw}), B(\psi_{slw}), C(\psi_{slw}), D(\psi_{slw}), E(\psi_{slw}) \): azimuthally dependent approximation coefficients (units of distance); in the case of pre-critical slowness match, the coefficients also depend on the matching slowness \( p_{h,f} \) and derivatives of these coefficients with respect to the slowness azimuth \( \psi_{slw} \)
- \( V_1 \): two-way average vertical velocity (azimuthally independent)
- \( V_2(\psi_{slw}) \): azimuthally dependent global second-order NMO velocity
- \( V_4(\psi_{slw}) \): azimuthally dependent global fourth-order NMO velocity
- \( M_2(\psi_{slw}) \): unitless azimuthally dependent second-order critical ray parameter
- \( M_4(\psi_{slw}) \): unitless azimuthally dependent fourth-order critical ray parameter
- \( S(\psi_{slw}) \): numerator of \( M_2 \)
- \( Q(\psi_{slw}) \): numerator of \( M_4 \)
- \( R(\psi_{slw}) \): common denominator of \( M_2 \) and \( M_4 \)

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APPENDIX A: AZIMUTHAL DERIVATIVES FOR CONSTANT NORMALIZED SLOWNESS

As shown in equation (21), we need to compute the azimuthal derivatives of the intercept time and its mixed derivatives, keeping the normalized slowness $p_b$ constant. This is not the same as the regular partial derivative $\frac{\partial \tau}{\partial \psi_{\text{slw}}}$, where the non-normalized horizontal slowness $p_b$ is kept constant. The reason is that the normalizing factor (the critical slowness) is azimuthally dependent, $p_c = p_c(\psi_{\text{slw}})$, hence

$$ p_b = \frac{p_b}{p_c(\psi_{\text{slw}})} \rightarrow p_b(\psi_{\text{slw}}) = p_b(p_b(\psi_{\text{slw}})). \tag{A1} $$

$$ \begin{align*}
\frac{\partial U(p_b, \psi_{\text{slw}})}{\partial \psi_{\text{slw}}} \bigg|_{p_b=\text{const}} &= \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial \psi_{\text{slw}}} \bigg|_{p_b=p_b(\psi_{\text{slw}})} = \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial \psi_{\text{slw}}} \bigg|_{p_b=\text{const}} \\
+ \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial p_b} \frac{\partial p_b(\psi_{\text{slw}})}{\partial \psi_{\text{slw}}} &= \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial \psi_{\text{slw}}} + \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial p_b} p_b(p_b(\psi_{\text{slw}})) \psi_{\text{slw}} \\
&= \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial \psi_{\text{slw}}} + \frac{\partial U(p_b, \psi_{\text{slw}})}{\partial p_b} p_b(p_b(\psi_{\text{slw}})) p_c(p_b(\psi_{\text{slw}})) \psi_{\text{slw}} \tag{A2} 
\end{align*} $$

Consider an arbitrary function $U(p_b, \psi_{\text{slw}})$, which may be in particular, but not necessarily, the intercept time. The azimuthal derivative for constant $p_b$ reads

Thus, the azimuthal derivative of the intercept time \( \tau \) reads

\[
\frac{\partial \tau (p_b, \psi_{slw})}{\partial \psi_{slw}} \bigg|_{p_b=\text{const}} = \frac{\partial \tau (p_b, \psi_{slw})}{\partial \psi_{slw}} + \frac{\partial \tau (p_b, \psi_{slw})}{\partial p_b} \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})}. \tag{A3}
\]

This equation may be also arranged as

\[
\frac{\partial \tau (p_b, \psi_{slw})}{\partial \psi_{slw}} \bigg|_{p_b=\text{const}} = -p_b \left[ b_T + b_R \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})} \right]. \tag{A4}
\]

This makes it possible to obtain the required mixed derivatives. Computing the derivative on the right-hand side of equation (A3) with respect to the non-normalized horizontal slowness \( p_b \), we obtain

\[
\frac{\partial^2 \tau (p_b, \psi_{slw})}{\partial p_b^2} \bigg|_{p_b=\text{const}} = \frac{\partial^2 \tau (p_b, \psi_{slw})}{\partial p_b \partial \psi_{slw}} + \frac{\partial \tau (p_b, \psi_{slw})}{\partial p_b} \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})} + \frac{\partial \tau (p_b, \psi_{slw})}{\partial \psi_{slw}} \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})}. \tag{A5}
\]

Repeating this operation, we obtain the higher mixed derivative,

\[
\frac{\partial^3 \tau (\psi_{slw}, p_b)}{\partial p_b^3} \bigg|_{p_b=\text{const}} = \frac{\partial^3 \tau (\psi_{slw}, p_b)}{\partial p_b \partial \psi_{slw}} + \frac{\partial^3 \tau (\psi_{slw}, p_b)}{\partial \psi_{slw}^2} \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})} + \frac{\partial \tau (p_b, \psi_{slw})}{\partial \psi_{slw}} \frac{p'_c (\psi_{slw})}{p_c (\psi_{slw})}. \tag{A6}
\]

Thus, to compute the coefficients of the pre-critical slowness match, we will need some, but not all, of the pure and mixed derivatives of the intercept time up to order 3 (higher derivatives with respect to the azimuth are not needed),

\[
\tau (\psi_{slw}, p_b), \frac{\partial \tau}{\partial p_b}, \frac{\partial^2 \tau}{\partial p_b^2}, \frac{\partial^2 \tau}{\partial \psi_{slw} \partial p_b}, \frac{\partial^3 \tau}{\partial \psi_{slw}^2 \partial p_b}, \frac{\partial^3 \tau}{\partial p_b \partial \psi_{slw} \partial p_b}, \frac{\partial^3 \tau}{\partial \psi_{slw}^3 \partial p_b}. \tag{A7}
\]

Equation (23) relates the global two-way intercept time \( \tau \) to the slowness surfaces of individual layers \( p^\psi_{b,i} \) and \( p^\psi_{c,i} \), corresponding to the wave types of the incidence and reflection rays. Thus, to compute the derivatives of the intercept time, we need the corresponding derivatives of the slowness surfaces. The latter can be computed analytically for any wave type by implicit differentiation of the Christoffel polynomial equation for an orthorhombic medium, as explained in Appendix B.

Since the order of differentiation in the mixed derivative of the intercept time \( \partial^2 \tau / (\partial p_b \partial \psi_{slw}) \) does not matter, we obtain one more useful relationship,

\[
b_T + p_b \frac{\partial b_c}{\partial p_b} = \frac{\partial b_R}{\partial p_b} = -\frac{\partial^2 \tau}{\partial p_b \partial \psi_{slw}}. \tag{A8}
\]

**APPENDIX B: DERIVATIVES OF THE SLOWNESS SURFACE**

The slowness surface is given by the bi-cubic polynomial equation (A3) in Part I, where the coefficients \( a, b, c, d \) of the slowness surface equation are functions of the horizontal slowness magnitude \( p_b \) and the azimuth \( \psi_{slw} \) (equations (A4) and (A8) of Part I). Computing successively the derivatives of the cubic polynomial, we obtain the derivatives of the slowness surface,

\[
\begin{align*}
\frac{\partial p_c}{\partial p_b} &= -1 - L_{bh} \frac{1}{2} + 3 b p_c^2 + 2 b p_b^2 + c, \\
\frac{\partial p_c}{\partial \psi_{slw}} &= -1 - L_{\psi} \frac{1}{2} + 3 b p_c p_b, \\
\frac{\partial^2 p_c}{\partial \psi_{slw}^2} &= -1 - L_{\psi \psi} \frac{1}{2} + 3 b p_c^2 + 2 b p_b^2 + c, \\
\frac{\partial p_c}{\partial p_b} &= -1 - L_{bh} \frac{1}{2} + 3 b p_c p_b, \\
\frac{\partial p_c}{\partial \psi_{slw}} &= -1 - L_{bh \psi} \frac{1}{2} + 3 b p_c p_b, \\
\frac{\partial^2 p_c}{\partial \psi_{slw}^2} &= -1 - L_{bh \psi \psi} \frac{1}{2} + 3 b p_c^2 + 2 b p_b^2 + c.
\end{align*}
\tag{B1}
\]

Parameters \( L_{bh}, L_{hh}, L_{hhb}, L_{bh \psi}, L_{bh \psi \psi} \) in the numerators are auxiliary functions related to corresponding derivatives. Subscript \( h \) means ‘related to the derivative of the slowness surface with respect to the horizontal slowness magnitude’, while subscript \( \psi \) means ‘... with respect to the slowness azimuth’. The auxiliary functions are

\[
\begin{align*}
L_{bh} &= \frac{\partial a}{\partial p_b} p_c^2 + \frac{\partial b}{\partial p_b} p_c^2 + \frac{\partial c}{\partial p_b} p_c^2 + \frac{\partial d}{\partial p_b}, \\
L_{hh} &= 2 \left( \frac{\partial p_c}{\partial p_b} \right)^2 (15 a p_c^4 + 6 b p_c^2 + c) \\
&\quad + 4 p_b \frac{\partial p_c}{\partial p_b} \left( \frac{\partial a}{\partial p_b} p_b^2 + \frac{\partial b}{\partial p_b} p_b^2 + \frac{\partial c}{\partial p_b} p_b^2 \right) \\
&\quad + \frac{\partial^2 a}{\partial p_b^2} p_c^2 + \frac{\partial^2 b}{\partial p_b^2} p_c^2 + \frac{\partial^2 c}{\partial p_b^2} p_c^2 + \frac{\partial^2 d}{\partial p_b^2}. \tag{B3}
\end{align*}
\]
\[ L_{shh} = \frac{\partial^3 a}{\partial p^3_b} p^2_b + \frac{3}{\partial p^3_b} \frac{\partial^3 c}{\partial p^3_b} p^2_c + \frac{\partial^3 d}{\partial p^3_b} \] 

\[ + 18 p^2_c \left( \frac{\partial^2 a}{\partial p^2_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^3 c}{\partial p^3_b} \right) + 12 p^2_c \left( \frac{\partial^2 b}{\partial p^2_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right) + 6 p_b \left( \frac{\partial c}{\partial p_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right) \]

\[ + 90 p^2_c \left[ \frac{\partial a}{\partial p_b} \left( \frac{\partial p}{\partial p_b} \right)^2 + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] + 36 p^2_c \left[ \frac{\partial b}{\partial p_b} \left( \frac{\partial p}{\partial p_b} \right)^2 + \frac{b}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] \]

\[ + 6 \left[ \frac{\partial c}{\partial p_b} \left( \frac{\partial p}{\partial p_b} \right)^2 + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] + 24 p_c \left( 5 \frac{\partial p}{\partial p_b} + b \right) \left( \frac{\partial p}{\partial p_b} \right)^3, \] (B4)

\[ L_{\psi} = \frac{\partial a}{\partial \psi_{slw}} p^2_b + \frac{\partial b}{\partial \psi_{slw}} p^2_c + \frac{\partial c}{\partial \psi_{slw}} p^2_c + \frac{\partial d}{\partial \psi_{slw}}, \] (B5)

\[ L_{\psi b} = \frac{\partial^2 a}{\partial p^2_b} \frac{\partial p}{\partial p_b} \] 

\[ + 4 p^2_b \left( \frac{\partial b}{\partial p_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial c}{\partial p_b} \right) + 2 \frac{\partial p}{\partial p_b} \frac{\partial^2 p}{\partial p^2_b} \]

\[ + 2 \frac{\partial^2 a}{\partial p^2_b} \frac{\partial p}{\partial p_b} \] 

\[ + 6 \frac{\partial c}{\partial p_b} \frac{\partial^2 p}{\partial p^2_b} \] 

\[ + 2 \left[ \frac{\partial^2 a}{\partial p^2_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] + 2 \left[ \frac{\partial^2 b}{\partial p^2_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] \]

\[ + 2 \left[ \frac{\partial^2 c}{\partial p^2_b} \frac{\partial p}{\partial p_b} + \frac{\partial a}{\partial p_b} \frac{\partial^2 c}{\partial p^2_b} \right] + 24 q \left( 5 \frac{\partial q}{\partial p} + b \right) \left( \frac{\partial q}{\partial p} \right)^2 \] (B7)

The formulae simplify slightly because coefficient \( a \) is independent of the magnitude and azimuth of the horizontal slowness. We need the derivatives of coefficients \( b, c, d \). These coefficients are listed in equation (A8) in Part 1, and their derivatives read

\[ \frac{\partial a}{\partial p_b} = 2 b_y p_b \cos^2 \Delta \psi_{slw} + 2 b_y p_b \sin^2 \Delta \psi_{slw}, \]

\[ \frac{\partial b}{\partial p_b} = 2 b_x \cos^2 \Delta \psi_{slw} + 2 b_y \sin^2 \Delta \psi_{slw}, \]

\[ \frac{\partial c}{\partial p_b} = 0. \] (B8)

\[ \frac{\partial a}{\partial \psi_{slw}} = (b_y - b_x) \frac{p_b}{p_b} \sin 2 \Delta \psi_{slw}, \]

\[ \frac{\partial b}{\partial \psi_{slw}} = 2 \left( b_x - b_y \right) \frac{p_b}{p_b} \sin 2 \Delta \psi_{slw}, \]

\[ \frac{\partial c}{\partial \psi_{slw}} = 2 \left( b_x - b_y \right) \sin 2 \Delta \psi_{slw}. \] (B9)

\[ \frac{\partial a}{\partial p_b} = 2 c_x p_b \cos^2 \Delta \psi_{slw} + 2 c_y p_b \sin^2 \Delta \psi_{slw} + 4 c_y p_b \cos^2 \Delta \psi_{slw} \]

\[ + 4 c_y p_b \sin^2 \Delta \psi_{slw} + 4 c_y p_b \cos^2 \Delta \psi_{slw} \sin^2 \Delta \psi_{slw}, \]

\[ \frac{\partial b}{\partial p_b} = 2 c_x \cos^2 \Delta \psi_{slw} + 2 c_y \sin^2 \Delta \psi_{slw} + 12 c_x p_b \cos^4 \Delta \psi_{slw} \]

\[ + 12 c_y p_b \sin^4 \Delta \psi_{slw} + 12 c_y p_b \cos^2 \Delta \psi_{slw} \sin^2 \Delta \psi_{slw}, \]

\[ \frac{\partial c}{\partial p_b} = 24 c_y p_b \cos^2 \Delta \psi_{slw} + 24 c_y p_b \sin^2 \Delta \psi_{slw}, \] (B10)
\[
\frac{\partial c}{\partial \psi_{\text{phs}}} = [(c_y - c_x) p_h^2 + (c_x - c_{xx}) p_h^4] \sin 2\Delta\psi_{\text{phs}}
- \frac{1}{2} (c_{xx} - c_{xy} + c_{yy}) p_h^6 \sin 4\Delta\psi_{\text{phs}},
\]
\[
\frac{\partial^2 c}{\partial p_h^2 \partial \psi_{\text{phs}}} = [2 (c_y - c_x) p_h + 4 (c_{yy} - c_{xx}) p_h^3] \sin 2\Delta\psi_{\text{phs}}
- 2 (c_{xx} - c_{xy} + c_{yy}) p_h^5 \sin 4\Delta\psi_{\text{phs}},
\]
\[
\frac{\partial^3 c}{\partial p_h^3 \partial \psi_{\text{phs}}} = [2 (c_y - c_x) + 12 (c_{yy} - c_{xx}) p_h^2] \sin 2\Delta\psi_{\text{phs}}
- 6 (c_{xx} - c_{xy} + c_{yy}) p_h^5 \sin 4\Delta\psi_{\text{phs}}. \quad (B11)
\]
\[
\frac{\partial d}{\partial p_h} = 2 d_{xx} p_h \cos^2 \Delta\psi_{\text{phs}} + 2 d_{yy} p_h \sin^2 \Delta\psi_{\text{phs}} + 4 d_{xxx} p_h^3 \cos^4 \Delta\psi_{\text{phs}}
+ 4 d_{xy} p_h^5 \sin^2 \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}}
+ 6 d_{xxx} p_h^5 \cos^2 \Delta\psi_{\text{phs}} + 6 d_{xyy} p_h^7 \sin^4 \Delta\psi_{\text{phs}}
+ 6 d_{xyy} p_h^7 \cos^4 \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}}
+ 6 d_{xyy} p_h^7 \cos^2 \Delta\psi_{\text{phs}} \sin^4 \Delta\psi_{\text{phs}}. \quad (B12a)
\]
\[
\frac{\partial^2 d}{\partial p_h^2} = 2 d_{xx} p_h \cos^2 \Delta\psi_{\text{phs}} + 2 d_{yy} p_h \sin^2 \Delta\psi_{\text{phs}} + 12 d_{xxx} p_h^5 \cos^4 \Delta\psi_{\text{phs}}
+ 12 d_{xy} p_h^7 \sin^4 \Delta\psi_{\text{phs}} + 12 d_{yx} p_h^7 \cos^4 \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}}
+ 30 d_{xxx} p_h^7 \cos^2 \Delta\psi_{\text{phs}} + 30 d_{xyy} p_h^9 \sin^6 \Delta\psi_{\text{phs}}
+ 30 d_{xyy} p_h^9 \cos^6 \Delta\psi_{\text{phs}} + 30 d_{xyy} p_h^9 \cos^2 \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}}
\times \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}} + 30 d_{xyy} p_h^9 \cos^2 \Delta\psi_{\text{phs}} \sin^4 \Delta\psi_{\text{phs}}. \quad (B12b)
\]
\[
\frac{\partial^3 d}{\partial p_h^3} = 24 d_{xx} p_h \cos^4 \Delta\psi_{\text{phs}} + 24 d_{yy} p_h \sin^4 \Delta\psi_{\text{phs}}
+ 24 d_{xxx} p_h^5 \cos^6 \Delta\psi_{\text{phs}} + 24 d_{xyy} p_h^7 \sin^8 \Delta\psi_{\text{phs}} + 120 d_{xxx} p_h^9 \cos^8 \Delta\psi_{\text{phs}}
+ 120 d_{xyy} p_h^9 \sin^{10} \Delta\psi_{\text{phs}} + 120 d_{xyy} p_h^9 \cos^8 \Delta\psi_{\text{phs}} \sin^2 \Delta\psi_{\text{phs}}
+ 120 d_{xyy} p_h^9 \cos^6 \Delta\psi_{\text{phs}} \sin^4 \Delta\psi_{\text{phs}}. \quad (B12c)
\]
\[
\frac{\partial d}{\partial \psi_{\text{phs}}} = [(d_y - d_x) p_h^2 + (d_{xy} - d_{xx}) p_h^4] \sin 2\Delta\psi_{\text{phs}}
- \frac{1}{2} (d_{xx} - d_{xy} + d_{yy}) p_h^6 \sin 4\Delta\psi_{\text{phs}}
- \frac{15 d_{xxx} + d_{axx} - d_{axx} - 15 d_{yy}}{16} p_h^6 \sin 2\Delta\psi_{\text{phs}}
- \frac{3 d_{xxx} - d_{axx} - d_{axx} + 3 d_{yy}}{4} p_h^6 \sin 4\Delta\psi_{\text{phs}}
- \frac{3 (d_{xxx} - d_{axx} + d_{axx} - d_{yy})}{16} p_h^6 \sin 6\Delta\psi_{\text{phs}}. \quad (B13a)
\]
\[
\frac{\partial^2 d}{\partial p_h \partial \psi_{\text{phs}}} = [2 (d_y - d_x) p_h + 4 (d_{xy} - d_{xx}) p_h^3] \sin 2\Delta\psi_{\text{phs}}
- 2 (d_{xx} - d_{xy} + d_{yy}) p_h^5 \sin 4\Delta\psi_{\text{phs}}
- \frac{3 (15 d_{xxx} + d_{axx} - d_{axx} - 15 d_{yy})}{8} p_h^6 \sin 2\Delta\psi_{\text{phs}}
\times \frac{3 (d_{xxx} - d_{axx} - d_{axx} + 3 d_{yy})}{2} p_h^6 \sin 4\Delta\psi_{\text{phs}}
- \frac{9 (d_{xxx} - d_{axx} + d_{axx} - d_{yy})}{8} p_h^6 \sin 6\Delta\psi_{\text{phs}}. \quad (B13b)
\]
\[
\frac{\partial^3 d}{\partial p_h^2 \partial \psi_{\text{phs}}} = [2 (d_y - d_x) + 12 (d_{xy} - d_{xx}) p_h^2] \sin 2\Delta\psi_{\text{phs}}
- 6 (d_{xx} - d_{xy} + d_{yy}) p_h^5 \sin 4\Delta\psi_{\text{phs}}
- \frac{15 (15 d_{xxx} + d_{axx} - d_{axx} - 15 d_{yy})}{8} p_h^6 \sin 2\Delta\psi_{\text{phs}}
- \frac{15 (3 d_{xxx} - d_{axx} - d_{axx} + 3 d_{yy})}{2} p_h^6 \sin 4\Delta\psi_{\text{phs}}
- \frac{45 (d_{xxx} - d_{axx} + d_{axx} - d_{yy})}{8} p_h^6 \sin 6\Delta\psi_{\text{phs}}. \quad (B13c)
\]
Coefficients \( a, b_x, b_y, c_x, c_y, c_{xx}, c_{yy}, c_{xy}, d_x, d_y, d_{xx}, d_{xy}, d_{xxx}, d_{xyy}, \) and \( d_{xyy} \) depend on the material stiffness tensor components. They are listed in equations (A4)–(A8) of Part I.

**APPENDIX C: A LIMIT CASE FOR THE PRE-CRITICAL SLOWNESS MATCH**

In this appendix, we study the limit case of the pre-critical slowness match, when the matching slowness approaches the critical value, \( p_h \to p_1(\psi_{\text{slw}}) \) or \( p_h \to 1 \). The five azimuthally dependent coefficients governing the approximations of the kinematical characteristics \( A(\psi_{\text{slw}}), B(\psi_{\text{slw}}), C(\psi_{\text{slw}}), D(\psi_{\text{slw}}), E(\psi_{\text{slw}}) \) result from the linear equation set (12). The corresponding matrix \( M \) depends on the normalized horizontal slowness \( p_h \) (equation (13)), and the right-hand side depends on both the horizontal slowness and the slowness azimuth \( \psi_{\text{slw}} \) (equation (14)). For the transverse offset component \( h_T \), we also need the azimuthal derivatives of the governing coefficients, which can be obtained from the linear set (19) with the same matrix and different right-hand side (equation (20)). The critical slowness match cannot be considered a particular case of the pre-critical slowness match by assuming \( p_h = 1 \), because the matrix components include the normalized matching complementary slowness \( \tilde{q}_f = \sqrt{1 - \tilde{p}_h^2} \) in
the denominator. The right-hand side includes unbounded values as well. Recall that the first two equations, for the second- and fourth-order NMO velocities, correspond to the normal-incidence ray. These equations are the same for the pre-critical and critical slowness matches. Thus, the limit operator has to be applied to the last three equations of the set: for the intercept time and its first and second derivatives with respect to the normalized horizontal slowness. Note that as we approach the critical matching slowness, only low-velocity layers contribute to the intercept time, because the vertical slowness in the high-velocity layer vanishes. However, only the high-velocity layer contributes to the derivatives of the intercept time with respect to the horizontal slowness, because the first derivative of the intercept time is the radial offset. At the critical point, the contribution of the low-velocity layers to the offset is bounded, while that of the high-velocity layer is unbounded and prevails. The two equations of the linear set for the coefficients of the critical slowness match, responsible for $M_z$ and $M_{ph}$, are (see equation set (2) of Part I),

\[
A(\psi_{slw}) + B(\psi_{slw}) + C(\psi_{slw}) = \sqrt{M_z(\psi_{slw})} \Delta z_{II}, \\
B(\psi_{slw}) + 2C(\psi_{slw}) = -\frac{3}{2} \frac{M_z(\psi_{slw})}{\sqrt{M_z(\psi_{slw})}} \Delta z_{II}, 
\]  

(C1)

where $\Delta z_{II}$ is the thickness of the high-velocity layer. However, we do not obtain both of them in the limit case of the pre-critical slowness match from the equations responsible for the first and second derivatives of the intercept time with respect to the horizontal slowness. Instead, we obtain the first equation of set (C1) twice.

Similarly, in the case of the critical slowness match, the last two equations of the linear set for the derivatives of coefficients $A'(\psi_{slw})$, $B'(\psi_{slw})$, $C(\psi_{slw})$, $D(\psi_{slw})$, $E'(\psi_{slw})$, responsible for $M_z$ and $M_{ph}$, are (see equation set (29) of Part I),

\[
A'(\psi_{slw}) + B'(\psi_{slw}) + C'(\psi_{slw}) = \frac{M_z(\psi_{slw})}{2\sqrt{M_z(\psi_{slw})}} \Delta z_{II} \\
B'(\psi_{slw}) + 2C'(\psi_{slw}) = -\frac{3}{4} \frac{M_z(\psi_{slw})}{\sqrt{M_z(\psi_{slw})}} \frac{M_z(\psi_{slw})}{\sqrt{M_z(\psi_{slw})}} \Delta z_{II} \\
-\frac{3}{2} \frac{M_z(\psi_{slw})}{\sqrt{M_z(\psi_{slw})}} \Delta z_{II}. 
\]  

(C2)

Again, we do not obtain both of them in the limit case of the pre-critical slowness match. Instead, we obtain the first equation of set (C2) twice.

These results are predictable. The equation for the slowness surface of the high-velocity layer (which is just the scaled intercept time) follows from equation (14) of Part I,

\[
p_{c, II} = \frac{\Delta \tau_{II}}{\Delta z_{II}}, \quad \frac{\Delta \tau_{II}}{\Delta z_{II} p_c(\psi_{slw})} = \sqrt{M_z} \frac{\bar{q}}{\Delta 1} + \frac{M_{ph}}{2\sqrt{M_z}} O(\bar{q}^4). 
\]  

(C3)

The first and second derivatives of the slowness surface read

\[
\frac{\Delta \tau_{II}'}{\Delta z_{II} p_c(\psi_{slw})} = -\frac{\sqrt{M_z}}{\bar{q}} + \frac{(M_z - 3M_{ph})}{2\sqrt{M_z}} \bar{q} + O(\bar{q}^3), \\
\frac{\Delta \tau_{II}''}{\Delta z_{II} p_c(\psi_{slw})} = -\frac{\sqrt{M_z}}{\bar{q}^2} + \frac{3}{2} \frac{M_{ph}}{\sqrt{M_z}} \bar{q} + O(\bar{q}). 
\]  

(C4)

Both equations of set (C4) include the leading term containing $M_z(\psi_{slw})$, a smaller term containing $M_{ph}(\psi_{slw})$ (or a combination of both $M_z$ and $M_{ph}$), and negligible high-order terms. As we approach the critical slowness $\bar{q} \rightarrow 0$, the first term with $M_z$ prevails in both equations, for the first and second derivatives. In the limit case, the contribution of the second term with $M_{ph}$ vanishes as compared with the contribution of the first term with $M_z$,

\[
\lim_{\bar{q} \rightarrow 0} \frac{\Delta \tau_{II}'}{\Delta z_{II} p_c(\psi_{slw})} = -\sqrt{M_z}(\psi_{slw}), \\
\lim_{\bar{q} \rightarrow 0} \frac{\Delta \tau_{II}''}{\Delta z_{II} p_c(\psi_{slw})} = -\sqrt{M_z}(\psi_{slw}). 
\]  

(C5)

As a result, in the limit case of the pre-critical slowness match, two equations of the set for either governing coefficients or their azimuthal derivatives become identical. Actually, in the limit case there are only four equations for five unknown variables, and the set becomes irresolvable. Thus, the limit case of the pre-critical slowness match does not converge to the critical slowness match. The reason is that the ‘pre-critical approach’ is based on matching the intercept time and its derivatives, while the ‘critical approach’ is based on the power series coefficients in the vicinity of the critical slowness. For the series with negative powers (normalized complementary slowness $\bar{q}$ in the denominator), these two approaches are not identical.

As we demonstrated in the numerical example for the multi-layer orthorhombic model, $p_0 = 0.999$ still leads to excellent results, exceeding the accuracy of the critical slowness match.

The resolving matrix of the pre-critical slowness match becomes non-invertible at the singular case $p_0 = 1$. In the critical slowness match approach, we directly study the case of the critical slowness, $p_0 = 1$. The resolving matrix in this case is invertible, and the solutions for both approximation coefficients and their azimuthal derivatives are unique.