Conic Velocity Model

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Summary

The effect of gradually increasing velocity with depth in compacted sediment layers is described by asymptotically bounded velocity models. These models can be defined by three intuitive parameters: the velocity and its vertical gradient at an initial depth level, and a bounded velocity value. We have recently introduced the exponential asymptotically bounded (EAB) model, which belongs to this family of velocity models. In this work we introduce another monotonously increasing and asymptotically bounded model, whose advantage is the simplicity of ray tracing. The ray trajectories in this model are elliptic, hyperbolic or parabolic. The linear velocity model with a circular trajectory is a particular case of the proposed model. Since the shape of the ray trajectories correspond to the three types of conic sections, we call the proposed velocity model conic.

Introduction

Recently we introduced the exponential asymptotically bounded (EAB) velocity model, (Ravve and Koren, 2004, 2006) described by three parameters: the velocity at the initial depth level \( V_a \), the vertical velocity gradient at the same level \( k_a \), and the asymptotic velocity at the infinite depth \( V_\infty \). This model is an extension of the EAB model. Among all possible asymptotically bounded models described by the same three parameters, the proposed conic model has two main advantages: a canonical form of ray trajectories and an explicit form of two-point ray tracing. In this work we first present the conic model and prove that the ray trajectories are elliptic, hyperbolic or parabolic. Next we describe the ray tracing procedure, and then derive the formulae for traveltime, lateral shift and arc length. Finally we consider a numerical example, comparing the EAB and the conic model.

Conic Model

Consider an infinite half-space, where the velocity vs. depth \( z \) is given by

\[
V(z) = \frac{1}{\sqrt{1 + Q^2(z + h)^2}} = \frac{R\tilde{z}}{\sqrt{1 + Q^2\tilde{z}^2}},
\]

where \( \tilde{z} = z + h \) is the “absolute” depth, and \( R, Q \) and \( h \) are three parameters that describe the model. The conic model can be also described in terms of the three parameters mentioned above: initial velocity \( V_a \), initial gradient \( k_a \) and asymptotic velocity \( V_\infty \).

\[
V_a = \frac{hR}{\sqrt{1 + h^2Q^2}}, \quad k_a = \frac{R}{(1 + h^2Q^2)^{1/2}}, \quad V_\infty = \frac{R}{Q}
\]

and the inverse relationships are

\[
R = \frac{k_a V_\infty^3}{V_\infty^2 - V_a^2}^{3/2}, \quad Q = \frac{R}{V_\infty}, \quad h = \frac{V_a^2 - V_\infty^2}{k_a V_\infty^2}.
\]

In particular, \( Q \to 0 \) corresponds to the linear velocity model with unbounded velocity. The velocity gradient decreases with depth until it becomes infinitesimal:

\[
k(z) = \frac{R}{1 + Q^2(z + h)^2}^{3/2}.
\]

For comparison, the EAB model is also a monotonously increasing and asymptotically bounded velocity function, described by

\[
V(z) = V_a + \Delta V_a \left[ 1 - \exp\left(-\frac{k_a z}{\Delta V}\right) \right], \quad V_\infty = V_a + \Delta V,
\]

where \( \Delta V \) is the instantaneous velocity range. Figure 1 shows an example of the EAB and the conic velocity models with the same values of three parameters.

Ray Trajectories

Establish the trajectories of non-vertical rays. In 1D medium the horizontal slowness \( p \) is constant, and the ray angle becomes: \( \sin \alpha = p V(z) \). Introduce parameter

\[
m = \frac{Q}{p R} = \frac{1}{p V_\infty} = \frac{1}{P},
\]

where \( P = p V_\infty \) is the normalized ray slowness. We will further show that \( m \) is the eccentricity of the ray trajectory. With equation 1, the ray angle becomes

\[
\sin \alpha = \frac{p R \zeta}{\sqrt{1 + m^2 p R^2 \zeta^2}},
\]

so that

\[
\tan \alpha = \pm \frac{p R \zeta}{\sqrt{1 - m^2 p^2 R^2 \zeta^2}} \frac{dx}{d\zeta},
\]

where \( m^2 = 1 - m^2 \), and parameter \( m^2 \) may be positive or negative. Integrating equation 8, we obtain

\[
x(\zeta) - x_c = \pm \frac{\sqrt{1 - m^2 p^2 R^2 \zeta^2}}{m^2 p R} - 1,
\]

where \( x_c \) is the constant of integration, or

\[
p^2 R^2 m^4 \cdot (x - x_c)^2 + p^2 R^2 m^2 \cdot \zeta^2 = 1.
\]

Equation 10 may be rearranged as

\[
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\]
where \( A^2 = \frac{1}{m^2 \cdot p^2 \cdot R^2} \), \( \pm B^2 = \frac{1}{m^2 \cdot p^2 \cdot R^2} \).  
(12)

Equation 11 describes a trajectory of an elliptic or hyperbolic type, with the eccentricity
\[
e = \sqrt{1 + B^2 / A^2} = m ,
\]
where the minus sign corresponds to an ellipse, and the plus sign to a hyperbola. Depending on the ray slowness \( p \), parameters \( A \) and \( B \) are semi-axes of an ellipse or a hyperbola. We distinguish three types of rays: pre-critical hyperbolic rays \( m > 1 \), post-critical elliptic rays \( 0 < m < 1 \), and critical parabolic rays \( m = 1 \). For a linear velocity model, \( m = 0 \) and all rays are post-critical with a round trajectory of constant curvature \( p \cdot R \). Integrating equation 8 for the critical case \( m = 1 \), we obtain a parabola \( x - x_c = Q z^2 / 2 \). The main advantage of the conic model is simplicity of the two-point ray tracing procedure. While the EAB model leads to a nonlinear equation with an unknown ray parameter, which in turn requires an iterative procedure, the conic model leads to an explicit solution.

### Lateral Shift, Traveltime and Arc Length

In a 1D medium, it is convenient to express the lateral shift \( x \), traveltime \( t_s \) and arc length \( s \) through the ray angle \( \alpha \) and angle-dependent gradient \( k(\alpha) \),
\[
x = \frac{1}{p} \int_a^\alpha \sin \alpha \, d\alpha \cdot k(\alpha), \quad t_s = \frac{1}{\alpha} \int_a^\alpha \frac{d\alpha}{\sin \alpha \cdot k(\alpha)}, \quad s = \frac{1}{p} \int_a^\alpha \frac{d\alpha}{k(\alpha)},
\]
where \( \alpha_0 \) and \( \alpha_0 \) are ray angels at the departure and the destination points, respectively. It follows from the Snell’s law and equations 1 and 4 that
\[
k(\alpha) = R \cdot \left[ 1 - m^2 \sin^2 \alpha \right] = R \Delta^2 .
\]
where \( \Delta^2 = 1 - m^2 \sin^2 \alpha \). The normalized lateral shift is
\[
p \cdot R \cdot x = \frac{\cos \alpha}{m^2 \cdot \Delta}.
\]
The normalized traveltime reads
\[
R t_s = -\frac{m^2 \cdot \cos \alpha}{m^2 \cdot \Delta} \cdot \arctanh(\cos \alpha / \Delta) \quad \text{for } m < 1
\]
\[
= \frac{m^2 \cdot \sin \alpha \cdot \cos \alpha}{\Delta} \quad \text{for } m > 1 .
\]
(17)

The normalized arc length \( \bar{s} = p \cdot R \cdot s \) is expressed through elliptic integrals of the first kind \( F \) and second kind \( E \). In case of elliptic trajectory, the eccentricity \( m < 1 \) and
\[
\bar{s} = E(\alpha, \Delta m) + F(\alpha, \Delta m) = \frac{m^2}{m^2 - \sin \alpha \cdot \cos \alpha} .
\]
(18)

In case of hyperbolic trajectory, the eccentricity \( m > 1 \) and
\[
\bar{s} = E(\alpha, 1/m) + F(\alpha, 1/m) = \frac{m^2}{m^2 - \sin \alpha \cdot \cos \alpha} .
\]
(19)

where the elliptic integrals are defined as
\[
F(\alpha, m) = \int_0^\alpha \frac{d\alpha}{\sqrt{1 - m^2 \cdot \sin^2 \alpha}}, \quad E(\alpha, m) = \int_0^\Delta d\alpha .
\]
(20)

In case of parabolic trajectory, \( m = 1 \), \( k = R \cos^2 \alpha \) and
\[
p \cdot R \cdot x = \frac{1}{2 \cos^2 \alpha} \cdot R t_s = \frac{1}{2 \cos^2 \alpha} + \ln \tan \alpha
\]
\[
p \cdot R \cdot s = \frac{1}{2} \cdot \left( \frac{\sin \alpha}{\cos \alpha} \right) \cdot \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) .
\]
(21)

### Two Point Ray Tracing

Given the location of the end points \( (x_S, z_S) \) and \( (x_R, z_R) \), find the parameters of the elliptic or hyperbolic trajectory. For two point ray tracing, it is suitable to work in a shifted frame. Depth \( z \) is now measured from the origin above the earth’s surface. At \( z = 0 \) all rays become vertical, so that the centers of ellipses or hyperbolas are located there. Thus, the shapes of rays are given by
\[
\frac{(x - x_c)^2}{A^2} + \frac{z^2}{B^2} = 1,
\]
\[
\frac{z^2}{B^2} - \frac{(x - x_c)^2}{A^2} = 1 ,
\]
(22)

where \( x_c \) is the unknown horizontal shift. At any point in the elliptic or hyperbolic trajectory, the ray angles are
\[
\sin \alpha = \frac{A^2 z^2}{\sqrt{A^4 \cdot z^4 + B^4 \cdot (x - x_c)^4}} .
\]
(23)

According to Snell’s law for 1D medium, at any two points of the trajectory the ratio between the sinuses of ray angles equals the ratio of the velocities at these points. For the end points of the trajectory
\[
\sin \alpha_S / \sin \alpha_R = V_S / V_R .
\]
(24)

Combining equations 23 and 24, we obtain:
\[
A^4 \cdot z^2_S + B^4 \cdot (x_S - x_c)^2 = \frac{z^2_R}{V^2_S} \cdot \frac{z^2_R}{V^2_R} .
\]
\[
A^4 \cdot z^2_S + B^4 \cdot (x_S - x_c)^2 = \frac{z^2_R}{V^2_S} \cdot \frac{z^2_R}{V^2_R} .
\]
(25)

An ellipse or hyperbola passes through the endpoints of the trajectory, and thus the semi-axes can be expressed from equation 22,
\[
A^2 = \frac{(x_S - x_c)^2 \cdot z^2_S - (x_R - x_c)^2 \cdot z^2_R}{z^2_R - z^2_S} ,
\]
\[
B^2 = \frac{(x_S - x_c)^2 \cdot z^2_S - (x_R - x_c)^2 \cdot z^2_R}{(x_S - x_c)^2 - (x_R - x_c)^2} .
\]
(26)

The plus sign corresponds to an ellipse, and the minus to a hyperbola. We assume here that depths \( z_S \) and \( z_R \) are
different and not too close. Introducing equation 26 into 25, we get

\[
\frac{(x_S - x_c)^2 - (x_R - x_c)^2}{(x_S - x_c)^2 - (x_R - x_c)^2} \frac{z_R^2 + \frac{z_R^2}{z_R^2 - z_c^2}}{z_R^2 + \frac{z_R^2}{z_R^2 - z_c^2}} k_R x_R - x_c)^2 = \frac{z_R^2}{z_R^2} V_S^2
\]

This is a quadratic equation for the unknown shift \( x_c \).

\[
W_A x_C^2 - 2W_B x_C + W_C = 0 ,
\]

where the coefficients \( W_A, W_B \) and \( W_C \) can be derived from equation 27 in a straightforward manner. We check the sign in equation 25 to find whether the root corresponds to an elliptic or a hyperbolic ray. The solution of equation 28 is unique. To discard the wrong root of the quadratic equation, we consider two cases, in particular: Linear velocity distribution (circular arc ray) and constant velocity (straight ray). We concluded that the sign before the square root should be plus. Our computational experience shows that the non-physical extra root corresponds to endpoints that belong to different branches of the hyperbola.

As one can see from equation 26, the method does not work for two points at the same depth (full elliptic arc of a post-critical ray), \( z_S = z_R = z_c \). In this case the procedure is different. Due to the symmetry, the center of the ellipse is located laterally just between the endpoints of the chord, and the lateral shift \( x_c \) reads

\[
2 x_c = x_R + x_S .
\]

The semi-axes of the ellipse, \( A \) and \( B \) should be established. One condition is that the ellipse passes through one of the endpoints, say \( x_S \). The second point drops out and gives no new equation through the symmetry. However, we assume that the curvature of the ellipse at the endpoint of the arc is equal to the curvature of the conic trajectory, and this leads to an additional equation. The curvature of an elliptic line \( \frac{d^2}{ds^2} \) reads

\[
\Lambda = \frac{z'^2}{(1 + z'^2)^{3/2}} = \frac{\frac{A^4 V^4}{A^4 z'^2 + B^4 (x_S - x_c)^2}}{B^4 (x_S - x_c)^2} .
\]

Normally, the curvature of an ellipse is considered negative, but we will need to swap the sign on the right side of equation 22 because ray angles are measured from the vertical axis, and thus, for a positive horizontal slowness, the ray angles increase. Combine equations 23 and 30:

\[
\Lambda_S = \frac{B^4 \sin^2 \alpha_S}{A^2 z'^2} .
\]

Due to the Snell’s law, in 1D medium the curvature of the ray path is

\[
\Lambda_S = k_S \sin \alpha_S / V_S ,
\]

where the gradient \( k_S \) is given by equation 4. Equations 31 and 32 result in

\[
\frac{B^4 \sin^2 \alpha_S}{A^2 z'^2} = \frac{k_S^2}{V_S} .
\]

Consider a set of three equations: 22 (for ellipse), 23 and 33 with three unknown variables \( A, B, \) and \( \alpha_S \). We eliminate the vertical semi-axis and the ray angle, and obtain a quadratic equation for the horizontal semi-axis,

\[
A^2 - (x_S - x_c)^2 + z^2 (x_S - x_c)^2 = A^2 \cdot V_S / k_S .
\]

To eliminate the extra root, we consider a linear case, where \( V_S \to \infty \) and elliptic trajectory becomes circular, and conclude that the sign before the square root is plus,

\[
A^2 = (x_S - x_c)^2 + z^2 V_S / 2 k_S + \frac{z^2 V_S^2}{4 k_S^2} + z (x_S - x_c)^2 V_S - k_S z
\]

We calculate the minor semi-axis \( B \) from equation 22.

**Numerical Example**

In the numerical example we compare the velocities and the ray trajectories for an EAB and a conic velocity model with the same values of parameters: the top velocity \( V_T = 3000 \text{m/s} \), the top gradient \( k_T = 1 \text{s}^{-1} \) and the asymptotic velocity \( V_S = 6000 \text{m/s} \). Figure 1 shows the instantaneous velocities, and Figure 2 shows the difference between the velocity of the EAB model and that of the conic model. The EAB reaches faster the asymptotic value. However, in both cases the velocity tends to the same asymptotic value at infinite depth. The velocity difference increases, reaches a maximum value at some depth, and then decreases and finally vanishes at large depth. Figure 3 shows the variation of vertical velocity gradient with depth. At a small depth, the EAB velocity increases faster, and its gradient is larger. Figure 4 shows the pre-critical, critical and post-critical ray trajectories for the two models. The critical angle is the same for both models, \( \alpha_c = \arcsin(V_T / V_S) = 30^\circ \). The pre-critical angle was \( 22.5^\circ \), and the post-critical angle was \( 37.5^\circ \). Since at small depth the EAB gradient is larger, then the curvature \( \Lambda = pk(c) \) of the ray path in the EAB medium is also larger. This phenomenon is demonstrated in Figure 4. Two columns on the right side of Figure 4 show the velocities of the conic and EAB models vs. depth. Figure 5 shows the traveltime vs. length of the curved ray path for the Conic model. The post-critical trajectory is the shallowest, and thus the slowest.

**Conclusions**

The importance of the asymptotically bounded functions in describing physical velocity models in compacted sediment
layers has already been studied and documented. In this work we presented a unique velocity function which leads to a simplistic and analytical ray tracing. The ray trajectories within the conic model are ellipses, hyperbolas or parabolas, defined by the major semi-axis, the eccentricity and the horizontal location of the center. The explicit form of the two-point ray-tracing solution makes the conic model attractive. In addition, the instantaneous velocity in the conic model approaches the asymptotic value slower than that in the EAB model with the same parameters. This evidence is in a closer agreement with physical observations.

References


EDITED REFERENCES
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References