Moveout approximation for horizontal transversely isotropic and vertical transversely isotropic layered medium. Part II: effective model‡

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ABSTRACT

We use residual moveouts measured along continuous full azimuth reflection angle gathers, in order to obtain effective horizontal transversely isotropic model parameters. The angle gathers are generated through a special angle domain imaging system, for a wide range of reflection angles and full range of phase velocity azimuths. The estimation of the effective model parameters is performed in two stages. First, the background horizontal transversely isotropic (HTI)/vertical transversely isotropic (VTI) layered model is used, along with the values of reflection angles, for converting the measured residual moveouts (or traveltime errors) into azimuthally dependent normal moveout (NMO) velocities. Then we apply a digital Fourier transform to convert the NMO velocities into azimuthal wavenumber domain, in order to obtain the effective HTI model parameters: vertical time, vertical compression velocity, Thomsen parameter delta and the azimuth of the medium axis of symmetry. The method also provides a reliability criterion of the HTI assumption. The criterion shows whether the medium possesses the HTI type of symmetry, or whether the azimuthal dependence of the residual traveltime indicates to a more complex azimuthal anisotropy. The effective model used in this approach is defined for a 1D structure with a set of HTI, VTI and isotropic layers (with at least one HTI layer). We describe and analyse the reduction of a multi-layer structure into an equivalent effective HTI model. The equivalent model yields the same NMO velocity and the same offset azimuth on the Earth’s surface as the original layered structure, for any azimuth of the phase velocity. The effective model approximates the kinematics of an HTI/VTI layered structure using only a few parameters. Under the hyperbolic approximation, the proposed effective model is exact.

Key words: Anisotropy, Modelling, Inversion, Parameter estimation, Velocity analysis.

INTRODUCTION

Acquired geophysical data often indicate a clear azimuthal dependency of the propagation velocity. The simplest model that describes a medium with azimuthally dependent properties is the horizontal transverse isotropy. It can be described by the vertical compression velocity, the azimuthal orientation of the axis of symmetry and Thomsen (1986) parameters. To establish the parameters of the effective horizontal transversely isotropic model, we use residual moveouts measured along continuous full azimuth reflection angle gathers. The angle gathers are generated through a special angle domain
imaging system (Koren et al. 2007, 2008), for a wide range of reflection angles \( \theta_{\text{phs}} \) and full range of phase velocity azimuths \( \varphi_{\text{phs}} \). The estimation of the effective model parameters is performed in two stages. First, the background horizontal transversely isotropic (HTI)/vertical transversely isotropic (VTI) layered model is used along with the values of reflection angles, for converting the measured residual moveouts (or traveltime errors) into azimuthally dependent residual normal moveout (NMO) velocities,

\[
\Delta t(\theta_{\text{phs}}, \varphi_{\text{phs}}) \rightarrow \Delta V_{\text{nmo}}(\varphi_{\text{phs}}).
\]  

(1)

Then we update the NMO velocity by adding the residual to the background value,

\[
V_{\text{nmo}}^{u}(\varphi_{\text{phs}}) = V_{\text{nmo}}^{b}(\varphi_{\text{phs}}) + \Delta V_{\text{nmo}}(\varphi_{\text{phs}}),
\]

(2)

where superscript ‘u’ stands for updated and superscript ‘b’ for background. The result is a number of series of NMO velocity versus phase velocity azimuth. The series differ by their reflection angle \( \theta_{\text{phs}} \), which is fixed for each series. In the second stage we apply a Fourier transform to convert each series of the NMO velocities into azimuthal wavenumber domain \( k_{\varphi} \),

\[
V_{\text{nmo}}^{2}(\varphi_{\text{phs}}) \rightarrow V_{\text{nmo}}^{2}(k_{\varphi}).
\]

(3)

Actually, we transform the updated NMO velocities squared. The NMO velocity in the Fourier domain is further used to obtain the effective HTI model parameters: the vertical time \( t_{0} \), the vertical compression velocity \( V_{v} \), Thomsen parameter \( \delta_{1} \) and the azimuth of the medium axis of symmetry \( \varphi_{\text{ax}} \). The method also provides a reliability criterion \( C \) of the HTI assumption. This criterion shows how close the azimuthal dependence of the traveltime is to that of an HTI medium. The effective model used in this approach is defined for a 1D structure with a set of HTI, VTI and isotropic layers with given properties and thickness and we assume that the package includes at least one HTI layer. We show that for studying the compression waves with near-vertical direction of propagation, this layered structure is equivalent to a unique homogeneous HTI layer, which can be considered as the effective model for the original layered structure. The equivalency means that the functions for NMO velocity versus the azimuth of the phase velocity, \( V_{\text{nmo}}^{u}(\varphi_{\text{phs}}) \), are the same for both the effective model and the original multi-layer model. In addition, the functions for the offset azimuth on the Earth’s surface versus the azimuth of the phase velocity, \( \varphi_{\text{eff}}(\varphi_{\text{phs}}) \), are also the same for the two models under the accuracy of the hyperbolic approximation. The effective model is valid for small offsets (i.e., near-vertical rays) and therefore it includes four parameters only: the thickness of the effective layer \( \theta_{\text{eff}}^{2} \), the effective vertical velocity \( V_{v}^{\text{eff}} \), the effective azimuth of symmetry \( \varphi_{\text{ax}}^{\text{eff}} \) and the effective Thomsen parameter \( \delta_{1}^{\text{eff}} \). The effect of parameter \( \varepsilon_{2} \) is ignored for small-offset rays. The derived effective model approximates the kinematics of an HTI/VTI layered structure using only a few parameters. Under the hyperbolic approximation, the proposed effective model is exact.

This paper is structured as follows. First we convert the residual moveout or traveltime error to the NMO velocity and then we show the method to estimate the effective model parameters from the NMO velocity function versus phase velocity azimuth. Next we define the effective model for a complex structure that includes HTI, VTI and isotropic flat layers and provide the relationships between the parameters of this model and the properties of individual layers. We provide a real data numerical example where effective models are established for different series of reflection angle. The detailed derivations have been moved to appendices. In Appendix A we explain the transformation of residual traveltime into residual moveout velocity and update the background NMO. In Appendix B we consider an important particular case when the background model is isotropic. Appendix C is devoted to Fourier analysis that enables the estimation of the effective model from the NMO velocity function. In Appendix D we further refine the parameters of the effective model to best fit the ‘measured’ NMO velocity. In Appendix E we explain the advantages of the effective model and derive its parameters, given the properties of the component layers. Finally, in Appendix F we derive the relationship of the NMO velocity versus the phase velocity azimuth starting from a similar relationship versus the ray (group) velocity azimuth.

### Normal Moveout Velocity from Full Azimuth Residual Moveouts

The normal moveout (NMO) velocity versus the azimuth of the phase velocity, \( V_{\text{nmo}}^{u}(\varphi_{\text{phs}}) \), can be calculated from residual moveouts (or traveltime errors), \( \Delta t(\theta_{\text{phs}}, \varphi_{\text{phs}}) \), specified versus the reflection angle and the azimuth of the phase velocity. We assume that the background effective model, horizontal transversely isotropic (HTI) or isotropic, is given and thus the background NMO velocity \( V_{\text{nmo}}^{b} \) can be computed for each value of the phase velocity azimuth \( \varphi_{\text{phs}} \).

\[
V_{\text{nmo}}^{b}(\varphi_{\text{phs}}) = \frac{1 + 4\delta_{1}^{b}(1 + \delta_{2}^{b}) \cos^{2}(\varphi_{\text{ax}}^{b} - \varphi_{\text{phs}})}{1 + 2\delta_{2}^{b} \cos^{2}(\varphi_{\text{ax}}^{b} - \varphi_{\text{phs}})}.
\]

(4)

The derivation of this formula is based on the work by Tsvankin (1997), where a similar relationship for the NMO
velocity versus the azimuth of the ray (group) velocity was obtained; see Appendix F for details. The updated NMO velocity \( V_{\text{nmo}}^u \), needed for the Fourier analysis, includes two counterparts: the background value and the residual; both items depend upon the phase velocity azimuth. The updated NMO velocity can be calculated by
\[
\frac{V_{\text{nmo}}^u(\varphi_{\text{phs}})}{V_{\text{nmo}}^b(\varphi_{\text{phs}})} = 1 - \frac{\xi(\varphi_{\text{phs}})}{\tan^2 \theta_{\text{phs}}} \frac{\Delta t(\theta_{\text{phs}}, \varphi_{\text{phs}})}{t_o},
\]
(5)

see Appendix A for details. The range of the phase velocity azimuth is \( 0 \leq \varphi_{\text{phs}} < \pi \). Function \( \xi(\varphi_{\text{phs}}) \) depends on the background model parameter \( \delta_2^b \) and the background azimuth of symmetry \( \varphi_{\text{ax}}^b \),
\[
\xi(\varphi_{\text{phs}}) = \frac{1}{1 + 2\delta_2^b \cos^2 (\varphi_{\text{ax}}^b - \varphi_{\text{phs}})}.
\]
(6)

Note that \( \delta_2^b \) is normally assumed negative and thus, \( \xi(\varphi_{\text{phs}}) \geq 1 \), where \( \xi = 1 \) corresponds to the isotropic background model. In other words, the HTI anisotropy amplifies the influence of residual traveltime on the residual NMO velocity, as compared to the isotropic model.

Next we consider a particular practical case when the background model is isotropic. In this case the background NMO velocity \( V_{\text{nmo}}^b \) is constant, i.e., independent on the azimuth of the phase velocity. Assume that the updated model differs only slightly from the background model, i.e., the anisotropy is weak. We introduce two dimensionless small parameters, the relative major and minor NMO velocities,
\[
\alpha_{\text{major}} = \frac{V_{\text{nmo}}^u - V_{\text{nmo}}^b}{V_{\text{nmo}}^b}, \quad \alpha_{\text{minor}} = \frac{V_{\text{nmo}}^b - V_{\text{nmo}}^h}{V_{\text{nmo}}^b}.
\]
(7)

With the assumptions on the isotropic background model and weak anisotropy in the updated model, Thomsen parameter delta becomes
\[
\delta_2 \approx \alpha_{\text{minor}} - \alpha_{\text{major}}.
\]
(8)

The details are presented in Appendix B. The residual traveltime depends on the major and minor NMO velocities,
\[
\frac{\Delta t(\theta_{\text{phs}}, \varphi_{\text{phs}})}{t_o} = -\tan^2 \theta_{\text{phs}} \left[ \alpha_{\text{major}} \sin^2 (\varphi_{\text{ax}} - \varphi_{\text{phs}}) + \alpha_{\text{minor}} \cos^2 (\varphi_{\text{ax}} - \varphi_{\text{phs}}) \right].
\]
(9)

A similar approach was used by Grechka and Tsvankin (1998), where the moveout varies versus the ray velocity azimuth. Note that equations (8) and (9) and Appendix B are the only parts in this study where the anisotropy is assumed weak; the degree of anisotropy is arbitrary elsewhere.

### EFFECTIVE MODEL PARAMETERS
### BY FOURIER ANALYSIS

In the case when the normal moveout (NMO) velocity versus phase velocity azimuth, \( V_{\text{nmo}}(\varphi_{\text{phs}}) \), is a periodic function with period \( \pi \), the medium may be a candidate for a horizontal transversely isotropic (HTI) model. In this section we describe the algorithm that makes it possible to estimate the parameters of effective model, applying the Fourier analysis to the NMO velocity squared. In addition, the method yields a consistency criterion. Note that not all \( \pi \)-periodic NMO velocity functions correspond to an HTI model exactly. There may be different azimuthally dependent NMO that correspond to more complex types of azimuthal anisotropy. The consistency criterion indicates how closely the input data, \( V_{\text{nmo}}(\varphi_{\text{phs}}) \), matches the HTI model. In this section we present the workflow of the Fourier analysis and the details of derivation are given in Appendix C. Assume that the NMO velocity is sampled on the uniform grid in the range \( 0 \leq \varphi_{\text{phs}} < \pi \). Assume also that the number of samples \( n \) is the Fast Fourier Transform (FFT) number; otherwise we resample the NMO values. The sampled values are
\[
V_{\text{nmo}, m} = V_{\text{nmo}}(\varphi_{\text{phs}, m}), \quad \varphi_{\text{phs}, m} = \frac{\pi m}{n}, \quad m = 0, 1, \ldots, n - 1.
\]
(10)

We apply a digital, real-to-complex Fourier transform to the NMO velocity squared and normalize the results by factor \( 1/n \) for DC and \( 2/n \) for AC,
\[
F_o = \frac{1}{n} \sum_{m=0}^{n-1} V_{\text{nmo}, m}^2, \quad \varphi_{\text{phs}, m} = \frac{\pi m}{n},
\]
(11)
\[
F_k = \frac{2}{n} \sum_{m=0}^{n-1} V_{\text{nmo}, m}^2 \exp \left( \frac{2\pi i km}{n} \right), \quad k = 1, 2, \ldots, N - 1,
\]
(12)

where \( N \) is the amount of complex numbers in the Fourier space,
\[
N = \frac{n}{2} + 1 \quad \text{for even } n, \quad N = \frac{n + 1}{2} \quad \text{for odd } n.
\]
(13)

The effective azimuth of the axis of symmetry reads
\[
\varphi_{\text{ax}}^{\text{eff}} = \frac{\arg (-F_1)}{2}, \quad -\frac{\pi}{2} < \varphi_{\text{ax}}^{\text{eff}} \leq \frac{\pi}{2}.
\]
(14)

The effective vertical velocity squared is delivered by
\[
V_{\text{ver}}^{\text{eff}} = F_o + \sum_{k=1}^{N-1} (-1)^{k+1} |F_k|.
\]
(15)
Thomsen parameter $\delta_{2}^{\text{eff}}$ can be obtained by
\[ \delta_{2}^{\text{eff}} V_{\text{ver}} = - \sum_{k=1,3,5} |F_k|. \]

To obtain the consistency criterion we compute two auxiliary values,
\[ \varphi_{\text{ax},m}^* = \text{arg} (-F_2) \pm \frac{2\pi m}{4}, \quad m = \{0, 1\}. \]

We use either plus or minus for $m = 1$ in equation (17) to get the result within the range specified in equation (14). The consistency criterion $C$ is the cosine of the angular difference between the azimuth of the symmetry axis and one of two auxiliary azimuths obtained in equation (17). The auxiliary azimuth closer to the axis of symmetry should be taken,
\[ C = \max \left[ \cos (\varphi_{\text{ax}} - \varphi_{\text{ax},1}^*), \cos (\varphi_{\text{ax}} - \varphi_{\text{ax},2}^*) \right]. \]

As one can see from equation (17), the two angles, $\varphi_{\text{ax},1}^*$ and $\varphi_{\text{ax},2}^*$ differ by $\pi/2$ and therefore at least one of the two cosines in equation (17) should be positive, so the range of the criterion is $0 < C \leq 1$. For the exact data that correspond to an HTTI model, or to a package of horizontal transversely isotropic, vertical transversely isotropic and isotropic layers, the consistency criterion $C = 1$. Otherwise, this criterion shows how noisy the data are or how different the medium, with the given azimuthal dependence of the NMO velocity, is from an HTTI model. The method described above is based on the Fourier analysis and it yields the NMO velocity in two azimuthal directions: along the azimuth of the effective symmetry axis and along the ‘isotropic’ azimuth, normal to the axis of symmetry. Alternatively, we can use the results listed from this analysis as an initial guess for the true effective parameters, minimizing

the discrepancy integral
\[ A = \frac{1}{2} \int_{\varphi_{\text{phs}}=0}^{\varphi_{\text{phs}}=\pi} \left[ V_{\text{nmo}} \left( V_{\text{ver}}^{\text{eff}}, \delta_{2}^{\text{eff}}, \varphi_{\text{ax}}, \varphi_{\text{phs}} \right) - V_{\text{data}} \left( \varphi_{\text{phs}} \right) \right]^2 \, d\varphi_{\text{phs}}, \]

where $V_{\text{nmo}}(V_{\text{ver}}^{\text{eff}}, \delta_{2}^{\text{eff}}, \varphi_{\text{ax}}, \varphi_{\text{phs}})$ is the calculated value of the moveout velocity, while $V_{\text{data}}(\varphi_{\text{phs}})$ are the ‘measured’ data. Although the calculated NMO velocity is always $\pi$-periodic, the measured value may have period $2\pi$ and in this case the upper limit of the integral becomes $2\pi$. As we mentioned, for a true horizontal transversely isotropic/vertical transversely isotropic layered structure, the initial guess leads to an exact solution. Otherwise, the initial guess is normally a good approximation and the non-linear minimization problem can be solved, see Appendix D for details. In many cases, however, the accuracy of the initial guess does suffice and the iterative procedure can be avoided.

**MEDIUM WITH CRACKS AND LAYERING**

A medium with both cracks and layering is orthorhombic, provided the cracks and the layers are mutually orthogonal. In a more general case with an arbitrary angle between the normal to the fracture plane and the normal to the layering plane, the medium may have even lower symmetry. A typical example would be a single set of dipping cracks in a vertical transversely isotropic background that leads to a monoclinic anisotropy with a single plane of symmetry, shown schematically in Fig. 1. In the scheme, $\alpha$ is the layering plane and $\beta$ is the fault plane and they are not necessarily orthogonal. Line $A$ is an intersection of the layering plane and the fault

\[ \gamma \text{ is plane of symmetry for monoclinic anisotropy,} \]
\[ \gamma \text{ is normal to plane } A \]
\[ \text{planes } \alpha \text{ and } \beta \text{ are not necessarily orthogonal} \]
\[ \gamma \perp \alpha, \gamma \perp \beta, \gamma \perp A \]

\[ \text{Figure 1 Scheme of monoclinic anisotropy caused by dipping cracks in VTI background.} \]
plane. The monoclinic plane of symmetry $\gamma$ is normal to line $A$ and therefore it is normal to both the layering plane $\alpha$ and the fault plane $\beta$. In this work, however, we do not study the anisotropic media with symmetry lower than transverse isotropy. We assume a 1D layered structure, such that in each layer (i.e., locally) there is either only a vertical transversely isotropic (VTI) shale, or only a horizontal transverse isotropy (HTI) caused by vertical cracks. We study the traveltime and the offset components for near-vertical rays (short offsets) and we show that for these purposes the layered structure may be replaced by a single equivalent HTI layer with effective vertical compression velocity, effective Thomsen parameter delta, effective azimuth of symmetry and thickness (vertical time). The equivalence is exact for an arbitrary anisotropy: Thomsen parameters of the layers are not necessarily small. However, the proposed effective model is valid for short offsets only. The relationship between the offset components and the traveltime makes it possible to establish the magnitude and the azimuth of the NMO velocity of the effective model.

**PARAMETERS OF EFFECTIVE MODEL VERSUS PROPERTIES OF LAYERS**

Given a layered structure consisting of horizontal transversely isotropic (HTI), vertical transversely isotropic (VTI) and isotropic layers (with at least one HTI layer), one can estimate the parameters of the equivalent effective layer. First, we assume that the vertical time is preserved, so that

$$ t^\text{eff}_0 = \sum_i t_{0,i}. $$

(20)

Next we calculate three auxiliary parameters: $W_x$, $W_y$ and $U$. Note that only HTI layers contribute to $W_x$ and $W_y$ and all layers contribute to parameter $U$,

$$ W_x = \sum_i \delta_{2,i} \cos 2\varphi_{ax,i} t_{0,i} V^2_{\text{ver},i}, $$

$$ W_y = \sum_i \delta_{2,i} \sin 2\varphi_{ax,i} t_{0,i} V^2_{\text{ver},i}, $$

(21)

$$ U = \sum_{i \in \text{HTI}} (1 + \delta_{2,i}) t_{0,i} V^2_{\text{ver},i} + \sum_{i \in \text{VTI}} (1 + 2\delta_i) t_{0,i} V^2_{\text{ver},i} + \sum_{i \in \text{ISO}} t_{0,i} V^2_{\text{ver},i}. $$

(22)

Assuming that the Thomsen parameter $\delta^\text{eff}_{2}$ is negative, the effective parameters are as follows. The azimuthal orientation of the symmetry axis is delivered by

$$ \cos 2\varphi_{ax}^\text{eff} = -\frac{W_x}{W}, \quad \sin 2\varphi_{ax}^\text{eff} = -\frac{W_y}{W}. $$

(23)

where

$$ W = \sqrt{W_x^2 + W_y^2}. $$

(24)

Note that parameters $U$ and $W$ are both positive. The effective vertical velocity is

$$ V^\text{eff}_{\text{ver}} = \sqrt{\frac{W + U}{C^2_{ax}}} $$

(25)

Finally, the effective Thomsen parameter is

$$ \delta^\text{eff}_{2} = -\frac{W}{W + U}. $$

(26)

The full derivation is given in Appendix E. We considered the data of Package 1, described in Table 1 of Part I and we obtained the following results: $t^\text{eff}_0 = 1.76$ s, $V^\text{eff}_{\text{ver}} = 2.948$ km/s, $\delta^\text{eff}_{2} = -0.1403$, $\varphi_{ax}^\text{eff} = 0.9222$ rad = 52.84$^\circ$.

**FIELD EXAMPLE**

Full azimuth reflection angle gathers were generated using the imaging method described in Koren et al. (2008) for an ocean-bottom seismometer (OBS) data set. Figure 2 (to the top) shows an example of such an angle gather, where the data traces are organized in five angle sectors. Each sector shares the same reflection angle (fixed half opening angle): 10, 15, 20, 25 and 30 degrees, respectively and contains 90 traces of phase velocity azimuths, ranging from 0–180 degrees. Consistent azimuthal moveout variations can be clearly identified at a depth level of 3800 m. The event under investigation is shown in Fig. 2 (to the bottom), with the automatically picked residual moveouts overlaid. Geologically, this area can be classified as a fractured carbonate layer. Assuming a horizontal transversely isotropic layered model, we apply our method to analyse the effective parameters from each reflection angle.

| Table 1 Parameters of effective HTI model versus reflection angle |
|---------------------|------------------|------------------|------------------|------------------|------------------|
| $\theta_{\text{phs}}$ | $V_{\text{min}}$ | $V_{\text{maj}}$ | $\delta^\text{eff}_2$ | $\varphi_{ax}$ | $C$          |
| 10$^\circ$        | 2120            | 2230            | -0.040            | 41$^\circ$     | 0.996        |
| 15$^\circ$        | 2135            | 2235            | -0.036            | 37$^\circ$     | 0.996        |
| 20$^\circ$        | 2151            | 2237            | -0.031            | 33$^\circ$     | 0.998        |
| 25$^\circ$        | 2169            | 2239            | -0.029            | 27$^\circ$     | 0.999        |
| 30$^\circ$        | 2180            | 2140            | -0.025            | 23$^\circ$     | 0.999        |

For each data series with a fixed reflection angle, we obtain a set of effective parameters.
sector independently. The resulting parameters are shown in Table 1, where

\[ v_{\text{major}} = v_{\text{ver}}^{\text{eff}}, \quad v_{\text{minor}} = v_{\text{ver}}^{\text{eff}} \sqrt{1 + 2 \delta_{\text{eff}}}, \quad -0.5 < \delta_{\text{eff}} < 0. \]

(27)

Although the obtained parameters at each sector are slightly different, the parameters can still be classified (grouped) to provide valuable trend values, especially for the azimuth of the axis of symmetry (the orientation of the fracture system).

CONCLUSIONS

We described a new method to compute effective horizontal transversely isotropic model parameters from full azimuth residual moveouts. The method exploits the input residual moveouts obtained directly as a function of the reflection angle and the azimuth of the phase velocity. For any package consisting of horizontal transversely isotropic (HTI), vertical transversely isotropic (VTI) and isotropic layers, with at least one HTI layer, an equivalent effective HTI layer exists that yields the same magnitude and direction on the normal moveout velocity versus the phase velocity direction. The effective parameters are the thickness (or vertical time), vertical velocity, the azimuth of symmetry axis and Thomsen parameter \( \delta_2 \). We developed a consistency criterion that indicates how closely the sampled data matches the HTI type of symmetry. For a true HTI/VTI 1D layered structure, the solution for effective parameters is ‘exact’ under the hyperbolic approximation.

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APPENDIX A. RESIDUAL NORMAL MOVEOUT VELOCITY VERSUS RESIDUAL TRAVELTIME

The algorithm that delivers the effective horizontal transverse isotropy (HTI) parameters requires the normal moveout (NMO) velocity function sampled versus azimuth of the phase velocity, \( V_{\text{nmo}}(\phi_{\text{phs}}) \). However, the available datum is the azimuthally dependent residual traveltime, \( \Delta t(\phi_{\text{phs}}) \). The (updated) NMO velocity may be considered consisting of two parts: that of the background model and the residual NMO,

\[
V_{\text{nmo}}(\phi_{\text{phs}}) = V_{\text{nmo}}^{b}(\phi_{\text{phs}}) + \Delta V_{\text{nmo}}(\phi_{\text{phs}}).
\]  

(A1)

The background NMO velocity can be obtained from the background HTI medium parameters,

\[
V_{\text{nmo}}^{b}(\phi_{\text{phs}})^2 = \frac{1 + 4 \delta_1^b (1 + \delta_2^b) \cos^2 (\phi_{\text{phs}} - \phi_{\text{phs}})}{1 + 2 \delta_1^b \cos^2 (\phi_{\text{phs}} - \phi_{\text{phs}})}.
\]  

(A2)

Equation (A2) for the NMO velocity versus azimuth of the phase velocity was proved in Part I, based on the analysis of the infinitesimal difference \( \pi/2 - \alpha_{\text{phs}} \), where the phase angle \( \alpha_{\text{phs}} \) is the angle between the phase velocity and the HTI medium axis of symmetry. An alternative derivation based on Tsvankin’s (1997) equation for the NMO velocity versus azimuth of the ray velocity is given in Appendix F.

The background medium parameters: the vertical velocity \( V_{\text{ver}}^b \), the orthorhombic Thomsen parameter \( \delta_2^b \) and the azimuth of the symmetry axis \( \phi_{\text{ax}}^b \) are assumed known in equation (A2). In particular, one can start from the isotropic background model with vanishing \( \delta_2^b \), in this case the axis azimuth \( \phi_{\text{ax}}^b \) does not matter and equation (A2) simplifies to

\[
V_{\text{nmo}}^{b}(\phi_{\text{phs}}) = V_{\text{ver}}^b = \text{const}.
\]  

(A3)

Therefore, to obtain the updated NMO velocity \( V_{\text{nmo}}(\phi_{\text{phs}}) \), we need to estimate the residual NMO, \( \Delta V_{\text{nmo}}(\phi_{\text{phs}}) \). In this appendix, we explain the method that makes it possible to establish the azimuthally dependent residual NMO, given the azimuthally dependent residual traveltime, \( \Delta t(\phi_{\text{phs}}) \rightarrow \Delta V_{\text{nmo}}(\phi_{\text{phs}}) \). We assume here that a fixed series of residual moveouts is being processed, i.e., for all azimuths \( \phi_{\text{phs}} \), the residual traveltimes \( \Delta t \) correspond to the same fixed reference reflection angle \( \theta_{\text{phs}} \). This angle (which is also the zenith angle of the phase velocity) is expected to be a small finite known value. We start from the definition of the NMO velocity. Recall that the NMO velocity yields the hyperbolic approximation for the traveltime,

\[
t^2 = t_0^2 + \frac{h^2}{V_{\text{nmo}}^2}.
\]  

(A4)

Equation (A4) approximates the residual traveltime as

\[
t \Delta t = - \frac{h^2 \Delta V_{\text{nmo}}}{V_{\text{nmo}}^2}.
\]  

(A5)

For the hyperbolic approximation, the offsets are assumed small, so that \( t \approx t_0 \) and equation (A5) reduces to

\[
\Delta t = - \frac{h^2}{V_{\text{ver}}^b} \frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2}.
\]  

(A6)

The ratio between the offset and the layer thickness reads

\[
b = \frac{h}{h_0} V_{\text{ver}}^b = \tan \theta_{\text{ray}}.
\]  

(A7)

According to equation (6) of Part I, the zenith angle of the phase velocity \( \theta_{\text{phs}} \) is given by,

\[
\sin \theta_{\text{phs}} = \frac{\cos \alpha_{\text{phs}}}{\cos (\phi_{\text{ax}} - \phi_{\text{phs}})}.
\]  

(A8)

Neglecting the high-order (non-hyperbolic) term in equation (E7) of Part I and taking into account that the deviation \( \Delta \phi_{\text{phs}} \) of the phase angle from the right angle may be presented as

\[
\Delta \phi_{\text{phs}} \approx \sin \Delta \phi_{\text{phs}} = \sin (\pi/2 - \phi_{\text{phs}}) = \cos \phi_{\text{phs}},
\]  

(A9)

we obtain the hyperbolic approximation for the zenith angle of the ray velocity,

\[
\sin \theta_{\text{ray}} = \frac{C_0 \cos \alpha_{\text{phs}}}{\cos (\phi_{\text{ax}} - \phi_{\text{phs}})}.
\]  

(A10)

This yields the relationship between the zenith angles of the phase and ray velocities,

\[
\sin \theta_{\text{ray}} / \sin \theta_{\text{phs}} = C_0,
\]  

(A11)

where, for the background model,

\[
C_0 = 1 + 4 \delta_2^b (1 + \delta_2^b) \cos^2 (\phi_{\text{ax}}^b - \phi_{\text{phs}}^b).
\]  

(A12)

For small zenith angles, the ratio of their sines may be replaced by the ratio of their tangents

\[
\frac{\tan \theta_{\text{ray}}}{\tan \theta_{\text{phs}}} \Rightarrow \frac{\sin \theta_{\text{ray}}}{\sin \theta_{\text{phs}}} = C_0.
\]  

(A13)
Introducing equation (A13) into (A7) gives,

$$\frac{\theta}{t_0} = \frac{b}{C_b} v_{\text{ver}} \tan \theta_{\text{phs}}. \quad (A14)$$

Introduction of equation (A14) into (A6) results in

$$\frac{\Delta t}{t_0} = - \frac{C_b^2 v_{\text{ver}}^2}{V_{\text{nmo}}^2} \times \frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2} \times \tan^2 \theta_{\text{phs}}. \quad (A15)$$

It follows from equation (A2) that

$$\frac{v_{\text{ver}}^2}{V_{\text{nmo}}^2} = 1 + 2 \frac{\delta}{\phi_{\text{phs}}} \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right). \quad (A16)$$

Introduction of equation (A16) into (A15) leads to

$$\frac{\Delta t}{t_0} = - \left[ 1 + 2 \frac{\delta}{\phi_{\text{phs}}} \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right) \right] \tan^2 \theta_{\text{phs}} \times \frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2}. \quad (A17)$$

Inverting equation (A17), we obtain

$$\frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2} = - \frac{1}{1 + 2 \frac{\delta}{\phi_{\text{phs}}} \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)} \frac{\Delta t/t_0}{\tan^2 \theta_{\text{phs}}}. \quad (A18)$$

provided the reference reflection angle does not vanish, \(\theta_{\text{phs}} \neq 0\) (i.e., the ray is near-vertical but not strictly vertical).

As we see, the method does not require keeping the reflection angle constant and geometry with the data points of different zeniths and azimuths may work, provided the zenith values are small. For an isotropic background model, equation (A18) simplifies to

$$\frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2} = - \frac{\Delta t/t_0}{\tan^2 \theta_{\text{phs}}}. \quad (A19)$$

With the notation

$$\xi(\phi_{\text{phs}}) \equiv \frac{1}{1 + 2 \frac{\delta}{\phi_{\text{phs}}} \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)}. \quad (A20)$$

the residual NMO velocity becomes

$$\frac{\Delta V_{\text{nmo}}}{V_{\text{nmo}}^2} = - \frac{\xi(\phi_{\text{phs}})}{\tan^2 \theta_{\text{phs}}} \frac{\Delta t}{t_0}, \quad (A21)$$

and the updated NMO velocity follows from equation (A1),

$$\frac{V_{\text{nmo}}^2}{V_{\text{nmo}}^2} = 1 - \frac{\xi(\phi_{\text{phs}})}{\tan^2 \theta_{\text{phs}}} \frac{\Delta t}{t_0}, \quad (A22)$$

where for the isotropic background model, \(\xi = 1\).

APPENDIX B. RESIDUAL TRAVELTIME VERSUS MAJOR AND MINOR NORMAL MOVEOUT VELOCITY

Assume an isotropic background medium with the constant normal moveout (NMO) velocity \(v_{\text{nmo}}\). Suppose that two parameters, \(\alpha_{\text{major}}\) and \(\alpha_{\text{minor}}\), are specified, related to the relative residual NMO velocity of the updated medium, with respect to the background medium,

$$\alpha_{\text{major}} = \frac{V_{\text{nmo}}^\text{major} - V_{\text{nmo}}}{v_{\text{nmo}}}, \quad \alpha_{\text{minor}} = \frac{V_{\text{nmo}}^\text{minor} - V_{\text{nmo}}}{v_{\text{nmo}}}. \quad (B1)$$

where the major and the minor NMO depend on the vertical velocity \(V_{\text{ver}}\) of the updated medium and the orthorhombic Thomsen parameter \(\delta_2\),

$$V_{\text{nmo}}^\text{major} = V_{\text{ver}}, \quad V_{\text{nmo}}^\text{minor} = V_{\text{ver}} \sqrt{1 + 2 \delta_2}. \quad (B2)$$

Recall that Thomsen parameter \(\delta_2\) is normally negative. Equation (B1) leads to

$$V_{\text{nmo}}^\text{major} = V_{\text{nmo}} (1 + \alpha_{\text{major}}), \quad V_{\text{nmo}}^\text{minor} = V_{\text{nmo}} (1 + \alpha_{\text{minor}}). \quad (B3)$$

The vertical velocity of the updated medium becomes

$$V_{\text{ver}} = V_{\text{nmo}} (1 + \alpha_{\text{major}}). \quad (B4)$$

The Thomsen parameter becomes

$$\sqrt{1 + 2 \delta_2} = \frac{V_{\text{nmo}}^\text{minor}}{V_{\text{nmo}}^\text{major}} = \frac{1 + \alpha_{\text{minor}}}{1 + \alpha_{\text{major}}}. \quad (B5)$$

and this leads to

$$\delta_2 = \frac{(1 + \alpha_{\text{minor}})^2 - (1 + \alpha_{\text{major}})^2}{2(1 + \alpha_{\text{major}})^2} = - \frac{(\alpha_{\text{major}} - \alpha_{\text{minor}})(2 + \alpha_{\text{minor}} + \alpha_{\text{major}})}{2(1 + \alpha_{\text{major}})^2}. \quad (B6)$$

The NMO velocity versus phase velocity azimuth is given by

$$\frac{V_{\text{ver}}^2(\phi_{\text{phs}})}{V_{\text{ver}}^2} = \frac{1 + 4 \delta_2 (1 + \delta_2) \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)}{1 + 2 \delta_2 \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)}. \quad (B7)$$

Introducing equations (B4) and (B6) into (B7) gives,

$$\frac{V_{\text{nmo}}^2(\phi_{\text{phs}})}{V_{\text{nmo}}^2} = \frac{(1 + \alpha_{\text{major}})^2 \sin^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right) + (1 + \alpha_{\text{minor}})^2 \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)}{(1 + \alpha_{\text{major}})^2 \sin^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right) + (1 + \alpha_{\text{minor}})^2 \cos^2 \left( \phi_{\text{ax}} - \phi_{\text{phs}} \right)}. \quad (B8)$$

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The relative residual NMO is defined as

\[
\frac{\Delta V_{nmo}(\psi_{phs})}{V_{nmo}} = \frac{V_{nmo}(\psi_{phs}) - V_{b_{nmo}}}{V_{b_{nmo}}}.
\]  

(B9)

Introduction of equation (B8) into (B9) leads to

\[
\Delta V_{nmo}(\psi_{phs}) = \frac{1 + \alpha_{major}^4 \sin^2(\phi_{ax} - \psi_{phs}) + (1 + \alpha_{major})^4 \cos^2(\phi_{ax} - \psi_{phs}) + (1 + \alpha_{major})^2 \cos^2(\phi_{ax} - \psi_{phs})}{1 + \alpha_{major}^2 \sin^2(\phi_{ax} - \psi_{phs}) + (1 + \alpha_{major})^2 \cos^2(\phi_{ax} - \psi_{phs})} - 1.
\]  

(B10)

So far, equation (B10) is valid for any degree of anisotropy. Now assume that the relative residuals of the NMO velocity are small,

\[
\alpha_{major} \ll 1, \quad \alpha_{minor} \ll 1.
\]  

(B11)

In this case equation (B6) simplifies to

\[
\delta_2 \approx \alpha_{minor} - \alpha_{major},
\]  

(B12)

i.e., the anisotropy is weak. Equation (B10) may be linearized for small values of \(\alpha_{minor}\) and \(\alpha_{major}\).

\[
\Delta V_{nmo}(\psi_{phs}) \approx \alpha_{major} \sin^2(\phi_{ax} - \psi_{phs}) + \alpha_{minor} \cos^2(\phi_{ax} - \psi_{phs}).
\]  

(B13)

Actually, the accuracy of the approximation depends only on condition (B12) and not (B11). We present three plots that compare exact and linearized solutions for the normalized residual NMO velocity in the phase plane (equations (B10) and (B13), respectively). In Fig. 3, the graph is plotted for major and minor residual NMO values of different signs: \(\alpha_{major} = 0.08, \alpha_{minor} = -0.06\). Blue lines correspond to the exact function and red lines to the linearized solution. Thick lines show the positive residual velocity and thin lines – the negative one. In Fig. 4, the relative residual NMO velocity is plotted for positive and essentially distinct major and minor values: \(\alpha_{major} = 0.12, \alpha_{minor} = 0.04\) (i.e., the anisotropy is large, \(\delta_2 \approx 0.08\). As we see, the residual NMO contour is concave. In Fig. 5, the relative residual NMO velocity is plotted for positive and close major and minor values: \(\alpha_{major} = 0.24, \alpha_{minor} = 0.18\). The anisotropy is weaker than in the previous case, \(\delta_2 \approx 0.06\) and the contour is convex. It follows from equation (B13) that the contour in the phase plane is convex when the major and minor residual NMO velocities are of the same sign and when they do not differ too much,

\[
\alpha_{major}/\alpha_{minor} < 3/2.
\]  

(B14)

From equation (A16), for the isotropic background model \(\xi = 1\), the relative residual traveltime is

\[
\Delta t \left(\theta_{phs}, \psi_{phs}\right)_{t_0} = -\tan^2 \theta_{phs} \frac{\Delta V_{nmo}(\psi_{phs})}{V_{b_{nmo}}}.
\]  

(B15)

We introduce equation (B13) into (B15) and obtain the relative residual traveltime in the phase plane,

\[
\Delta t \left(\theta_{phs}, \psi_{phs}\right)_{t_0} = -\tan^2 \theta_{phs} \left[\alpha_{major} \sin^2(\phi_{ax} - \psi_{phs}) + \alpha_{minor} \cos^2(\phi_{ax} - \psi_{phs})\right].
\]  

(B16)

APPENDIX C. HORIZONTAL TRANSVERSELY ISOTROPIC EFFECTIVE PARAMETERS BY FOURIER ANALYSIS

Assume that the normal moveout (NMO) velocity \(V_{nmo}(\psi_{phs})\) is sampled on a uniform azimuth grid. Our objective is to establish the parameters of the effective model. For a single horizontal transversely isotropic (HTI) layer (e.g., for the effective model), the NMO velocity versus the azimuth of the phase velocity reads,

\[
\frac{V_{nmo(\psi_{phs})}}{V_{ver}} = \frac{1 + 4 \delta_2 (1 + \delta_2) \cos^2(\phi_{ax} - \psi_{phs})}{1 + 2 \delta_2 \cos^2(\phi_{ax} - \psi_{phs})} = f^2(\psi_{phs}).
\]  

(C1)

This relationship may also be rewritten as

\[
\frac{V_{nmo(\psi_{phs})}}{V_{ver}} = 1 + 2 \delta_2 (1 + \delta_2) \cos^2(\phi_{ax} - \psi_{phs}) + 1 + 2 \delta_2 \cos^2(\phi_{ax} - \psi_{phs}) = f^2(\psi_{phs}).
\]  

(C2)

This is a periodic function of the phase velocity and since the cosines appear squared in the equation, then the period is \(\pi\) (and not \(2\pi\)). Assume that the data are sampled on the interval \([0, \pi]\) starting from an arbitrary reference (zero) azimuth. To study the behaviour of function on the right-hand side of equation (C2), we expand \(f^2\) into the Fourier series,

\[
\frac{V_{nmo(\psi_{phs})}}{V_{ver}} = A_0 + \sum_{k=1}^{\infty} A_k \cos(2k\psi_{phs}) + \sum_{k=1}^{\infty} B_k \sin(2k\psi_{phs}),
\]  

(C3)

where the coefficients of the expansion are delivered by

\[
A_k = \frac{1}{\pi} \int_{0}^{\pi} f^2(\psi_{phs}) d\psi_{phs},
\]  

(C4)
Figure 3 Relative residual NMO velocity in the phase plane, with major and minor values of different signs: $\alpha_{\text{major}} = 0.08$, $\alpha_{\text{minor}} = -0.06$. Blue lines – exact function, red lines – linearization; thick lines – positive residual velocity, thin lines – negative.

Figure 4 Relative residual NMO velocity in the phase plane, with positive major and minor values: $\alpha_{\text{major}} = 0.12$, $\alpha_{\text{minor}} = 0.04$. Blue lines – exact function, red lines – linearization. Concave contour, $\alpha_{\text{major}}/\alpha_{\text{minor}} > 3/2$.

Performing the integration, we obtain,

\[
A_k = \frac{2}{\pi} \int_0^{\pi} f^2(\phi_{\text{phs}}) \cos(2k\phi_{\text{phs}}) d\phi_{\text{phs}},
\]

and

\[
B_k = \frac{2}{\pi} \int_0^{\pi} f^2(\phi_{\text{phs}}) \sin(2k\phi_{\text{phs}}) d\phi_{\text{phs}}.
\]

Performing the integration, we obtain,

\[
A_k = M_k \cos(2k\phi_{\text{ax}}), \quad B_k = M_k \sin(2k\phi_{\text{ax}}).
\]
where

$$M_k = \frac{(-1)^{k+1} \cdot 2^{k+1} \delta^2}{(1 + \sqrt{1 + 2\delta^2})^k}, \quad k = 1, 2, \ldots$$  \hspace{1cm} (C8)

Alternatively, coefficients $A_k$ and $B_k$ can be presented as real and imaginary parts, respectively, of a complex number,

$$A_k = \text{Re} \hat{M}_k, \quad B_k = \text{Im} \hat{M}_k,$$  \hspace{1cm} (C9)

where

$$\hat{M}_k = M_k \exp(2i k \varphi_{ax}).$$  \hspace{1cm} (C10)

We note that coefficients $\hat{M}_k$ make an infinite geometric series with decreasing absolute values,

$$\frac{\hat{M}_{k+1}}{\hat{M}_k} = -\frac{2\delta^2 \exp(2i \varphi_{ax})}{(1 + \sqrt{1 + 2\delta^2})^k}. $$  \hspace{1cm} (C11)

Since $\delta^2$ is normally negative, the ratio $M_{k+1}/M_k$ is normally positive.

In Fig. 6 we plot the normalized NMO velocity versus ray velocity azimuth (blue line) and versus phase velocity azimuth (red line), for a single HTI layer (or for the equivalent layer of the package), for Thomsen parameter, $\delta^2 = -0.25$. The normalizing factor is the vertical velocity $V_{vee}$. The graph is plotted for the fixed azimuthal orientation of the HTI axis of symmetry, $\varphi_{ax} = 0$. The green line is the approximation to the NMO velocity versus phase velocity azimuth with the first Fourier harmonic. It yields an essential error. The grey line is the approximation with two Fourier harmonics. It is already very close to the exact value. The approximation with three harmonics (thin black line) is so accurate that it can not be distinguished from the exact graph.

Now assume that the Fourier coefficients are given or can be calculated and the parameters of the effective model are to be found. Again, we need to make a choice, whether we are looking for a positive or a negative $\delta$. Assume $\delta$ is negative. In this case, it follows from equation (C8) that all $M_k$, odd and even, are negative,

$$M_k = -\sqrt{A_k^2 + B_k^2}. $$  \hspace{1cm} (C12)

The azimuth of the axis of symmetry can be obtained from the first pair, $A_1$ and $B_1$,

$$2\varphi_{ax} = \arctan (-B_1, -A_1). $$  \hspace{1cm} (C13)

Note that parameter $M_k$ in equation (C7) is negative and this leads to minus signs in equation (C13). This solution is unique (within the assumption about negative $\delta$). The second pair gives two solutions,

$$4\varphi_{ax} = \arctan (-B_l, -A_l) + 2\pi m, \quad m = \{0, 1\}. $$  \hspace{1cm} (C14)

and one of these solutions (the proper solution) should coincide with the solution (C13). An arbitrary pair $A_l$, $B_l$ gives $l$ solutions,

$$2l\varphi_{ax} = \arctan (-B_l, -A_l) + 2\pi m, \quad m = \{0, 1, \ldots, l - 1\}. $$  \hspace{1cm} (C15)
Figure 6 NMO velocity versus azimuth of ray and phase velocity.

Only one of these solutions has a physical sense and for the non-noisy data obtained from the true horizontal transversely isotropic (HTI)/vertical transversely isotropic (VTI) layered structure, it coincides with the unique solution (C13). We exploit this property to check whether the medium under consideration possesses the HTI type of symmetry. After the azimuth of the symmetry axis is established, we can combine equations (C3) and (C7),

\[
\frac{V_{\text{nmo}}^2}{V_{\text{ver}}^2} = A_z + \sum_{k=1}^{\infty} M_k \cos (2k\varphi_{\text{phs}}) \cos (2k\varphi_{\text{ax}}) + \sin (2k\varphi_{\text{phs}}) \sin (2k\varphi_{\text{ax}}),
\]

(C16)

or equivalently,

\[
\frac{V_{\text{nmo}}^2}{V_{\text{ver}}^2} = A_z + \sum_{k=1}^{\infty} M_k \cos 2k(\varphi_{\text{phs}} - \varphi_{\text{ax}}). \tag{C17}
\]

Next, we consider two specific values of the phase velocity azimuth,

\[
\varphi_{\text{phs}} = \varphi_{\text{ax}}, \quad \frac{V_{\text{nmo}}^2}{V_{\text{ver}}^2} = A_z + \sum_{k=1}^{\infty} M_k, \tag{C18}
\]

and

\[
\varphi_{\text{phs}} = \varphi_{\text{ax}} + \pi/2, \quad \frac{V_{\text{nmo}}^2}{V_{\text{ver}}^2} = A_z + \sum_{k=1}^{\infty} M_k \cos k\pi = A_z + \sum_{k=1}^{\infty} (-1)^k M_k. \tag{C19}
\]

The first azimuth defines the vertical plane of symmetry and the second one defines the isotropic vertical plane. Both cases, (C18) and (C19), lead to an infinite decreasing geometric series: one with items of the same sign and another with items of alternating signs. Introducing \(M_k\) from equation (C8), we obtain

\[
A_z + \sum_{k=1}^{\infty} M_k = 1 + 2\delta_2, \quad A_z + \sum_{k=1}^{\infty} (-1)^k M_k = 1, \tag{C20}
\]

so that

\[
\delta_2 = \frac{1}{2} \sum_{k=1}^{\infty} M_k - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k M_k = \sum_{k=1,3,5} \infty M_k. \tag{C21}
\]

In practice, we deal with the NMO velocity squared, which is not scaled by the vertical velocity squared. In other words, we deal with \(V_{\text{nmo}}^2\) rather than with \(V_{\text{nmo}}^2/V_{\text{ver}}^2\). Thus, all the coefficients discussed obtain factor \(V_{\text{ver}}^2\). To get the Fourier coefficients, we apply the discrete forward Fourier transform to the NMO velocity squared, on the interval \(0 \leq \varphi_{\text{phs}} < \pi\), real-to-complex. Next, we normalize the DC value by factor \(1/n\) and the other (AC) values by factor \(2/n\),

\[
F_o = \frac{1}{n} \sum_{m=0}^{n-1} V_{\text{nmo}}^2 (\varphi_{\text{phs},m}), \quad \varphi_{\text{phs},m} = \frac{\pi m}{n}, \tag{C22}
\]

and

\[
F_k = \frac{2}{n} \sum_{m=0}^{n-1} V_{\text{nmo}}^2 (\varphi_{\text{phs},m}) \times \exp \left(\frac{2\pi i km}{n}\right), \quad k = 1, 2, \ldots, N - 1. \tag{C23}
\]
Note that normally the standard forward Fast Fourier Transform has a minus sign at the exponent in equation (C23). In this case one should invert the sign of all imaginary parts. Thus, we obtain the DC term and the AC terms,

\[ F_o = A_k V_{ver}^2, \quad \text{Re}(F_k) = A_k V_{ver}^2, \quad \text{Im}(F_k) = B_k V_{ver}^2. \]  

(C24)

The absolute AC values are

\[ |F_k| = -M_k V_{ver}^2. \]  

(C25)

Next it follows from equation (B20) that

\[ V_{ver}^2 = F_o - \sum_{k=1}^{N-1} (-1)^k |F_k|. \]  

(C26)

\( N \) is the number of complex items in the Fourier space of real-to-complex digital transform. This number includes the DC term and in case of even \( n \) also the Nyquist term. These two terms are real, while the others are complex,

\[ N = \frac{n}{2} + 1 \quad \text{for even} \ n, \quad N = \frac{n + 1}{2} \quad \text{for odd} \ n. \]  

(C27)

In practice, the absolute values \( |F_k| \) decay very quickly and 3–5 AC terms normally suffice. The orthorhombic Thomsen parameter \( \delta \) can be obtained from equation (C21),

\[ \delta V_{ver}^2 = -\sum_{k=1,3,5} |F_k|. \]  

(C28)

The values for effective parameters obtained in this section may be considered as an initial guess and further refined to best fit the ‘measured’ function of the NMO velocity versus the phase velocity azimuth, as shown in Appendix D.

APPENDIX D. REFINEMENT OF EFFECTIVE PARAMETERS

We solve equation (19) to find the effective horizontal transversely isotropic parameters that best fit the given data for the normal moveout (NMO) velocity versus the azimuth of the phase velocity. Combining equations (19) and (C1), we obtain

\[ A(V_{ver}^{eff}, \delta^{eff}, \varphi_{ax}^{eff}) = \frac{1}{2} \int_{\varphi_{phs}=0}^{\varphi_{phs}=\pi} \left[ \frac{V_{ver}^{eff} \times f(\delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs}) - V_{data}^{data}(\varphi_{phs})}{V_{data}^{data}(\varphi_{phs})} \right]^2 d\varphi_{phs}, \]

\[ A \rightarrow \min. \]  

(D1)

According to equation (C1), function \( f \) is defined by

\[ f^2 \left( \delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs} \right) = \frac{1 + 4 \delta^{eff} (1 + \delta^{eff}) \cos^2 (\varphi_{ax}^{eff} - \varphi_{phs})}{1 + 2 \delta^{eff} \cos^2 (\varphi_{ax}^{eff} - \varphi_{phs})}. \]  

(D2)

The NMO velocity depends on the effective vertical velocity linearly and this makes it possible to eliminate the effective vertical velocity,

\[ \frac{\partial A}{\partial V_{ver}^{eff}} = 0 \rightarrow V_{ver}^{eff}(\delta^{eff}, \varphi_{ax}^{eff}) = \frac{D(\delta^{eff}, \varphi_{ax}^{eff})}{B(\delta^{eff}, \varphi_{ax}^{eff})}. \]  

(D3)

where the coefficients \( B \) and \( D \) are delivered by

\[ D(\delta^{eff}, \varphi_{ax}^{eff}) \equiv \int_0^{\pi} \frac{f(\delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs})}{V_{data}^{data}(\varphi_{phs})} d\varphi_{phs}. \]

\[ B(\delta^{eff}, \varphi_{ax}^{eff}) \equiv \int_0^{\pi} \frac{f^2(\delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs})}{V_{data}^{data}(\varphi_{phs})} d\varphi_{phs}. \]  

(D4)

The effective vertical velocity \( V_{ver}^{eff} \) is no more an independent parameter and equation (D1) reduces to

\[ A(\delta^{eff}, \varphi_{ax}^{eff}) = \frac{1}{2} \int_{\varphi_{phs}=0}^{\varphi_{phs}=\pi} M^2(\delta^{eff}, \varphi_{ax}^{eff}, \delta^{eff}, \varphi_{ax}^{eff}) d\varphi_{phs} \rightarrow \min, \]  

(D5)

where

\[ M(\delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs}) \equiv \frac{V_{ver}^{eff}(\delta^{eff}, \varphi_{ax}^{eff}) \times f(\delta^{eff}, \varphi_{ax}^{eff}, \varphi_{phs}) - V_{data}^{data}(\varphi_{phs})}{V_{data}^{data}(\varphi_{phs})}. \]  

(D6)

To find the minimum, we solve a set of two non-linear equations with two unknown values,

\[ \frac{\partial A}{\partial \delta^{eff}} = \int_{\varphi_{phs}=0}^{\varphi_{phs}=\pi} \left[ \frac{\partial V_{ver}^{eff}}{\partial \delta^{eff}} \times f + V_{ver}^{eff} \times \frac{\partial f}{\partial \delta^{eff}} \right] \frac{M d\varphi_{phs}}{V_{data}^{data}(\varphi_{phs})} = 0, \]

\[ \frac{\partial A}{\partial \varphi_{ax}^{eff}} = \int_{\varphi_{phs}=0}^{\varphi_{phs}=\pi} \left[ \frac{\partial V_{ver}^{eff}}{\partial \varphi_{ax}^{eff}} \times f + V_{ver}^{eff} \times \frac{\partial f}{\partial \varphi_{ax}^{eff}} \right] \frac{M d\varphi_{phs}}{V_{data}^{data}(\varphi_{phs})} = 0. \]  

(D7)

The equation set can be solved effectively by the standard Newton method, applying the initial guess for the ‘exact’ layered structure. On each iteration, the corrections, \( \Delta \delta^{eff} \) and \( \Delta \varphi_{ax}^{eff} \), to the unknown parameters, can be found from the linear set

\[
\begin{bmatrix}
\frac{\partial^2 A}{\partial \delta^{eff}^2} & \frac{\partial^2 A}{\partial \delta^{eff} \partial \varphi_{ax}^{eff}} \\
\frac{\partial^2 A}{\partial \varphi_{ax}^{eff} \partial \delta^{eff}} & \frac{\partial^2 A}{\partial \varphi_{ax}^{eff}^2}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta^{eff} \\
\Delta \varphi_{ax}^{eff}
\end{bmatrix}
= -
\begin{bmatrix}
\frac{\partial A}{\partial \delta^{eff}} \\
\frac{\partial A}{\partial \varphi_{ax}^{eff}}
\end{bmatrix}.
\]  

(D8)
APPENDIX E. EFFECTIVE MODEL FOR LAYERED STRUCTURE

We proved in Part I that the normal moveout velocity of a package of horizontal transversely isotropic (HTI) and vertical transversely isotropic (VTI) layers is given by

\[ V_{nmo}^2 = \frac{A_{ph}^2}{2b_i A_{pt}}, \]  

(E1)

where

\[ A_{ph}^2 = A_{px}^2 + A_{py}^2. \]  

(E2)

Parameters \( A_{px} \) and \( A_{py} \) describe the lateral propagation in the direction of the phase velocity azimuth and in the normal direction, respectively, while parameter \( A_{ph} \) describes the traveltime. Recall that parameters \( A_{px}, A_{py} \) and \( A_{ph} \) are obtained by summation over all layers: HTI, VTI and isotropic but only HTI layers contribute into \( A_{py} \),

\[ A_{px} = \sum_i A_{px, i}^{HTI} + \sum_i A_{px, i}^{VTI} + \sum_i A_{px, i}^{ISO}, \quad A_{py} = \sum_i A_{py, i}^{HTI}, \]  

(E3)

\[ A_{pt} = \sum_i A_{pt, i}^{HTI} + \sum_i A_{pt, i}^{VTI} + \sum_i A_{pt, i}^{ISO}. \]  

(E4)

For an HTI layer, the offset coefficients are

\[ A_{px, i}^{HTI} = C_{b, i} \cos \Delta \varphi_{o,i} t_{o,i} V_{ver,i}^2, \quad A_{py, i}^{HTI} = C_{b, i} \sin \Delta \varphi_{o,i} t_{o,i} V_{ver,i}^2, \]  

(E5)

where \( \Delta \varphi_{o,i} \) is the azimuthal shift between the ray and the phase velocities for an infinitesimal offset, within the hyperbolic approximation,

\[ \cos \Delta \varphi_{o,i} = \frac{1 + 2 \delta_{2, i} \cos^2 (\varphi_{ax,i} - \varphi_{ph}),}{C_{b, i}} \]  

(E6)

\[ \sin \Delta \varphi_{o,i} = \frac{\delta_{2, i} \sin 2 (\varphi_{ax,i} - \varphi_{ph})}{C_{b, i}}. \]  

(E7)

Parameter \( C_b \) is defined by

\[ C_b = \sqrt{1 + 4 \delta_2 (1 + \delta_2) \cos^2 (\varphi_{ax} - \varphi_{ph})}, \]  

(E8)

and the traveltime coefficient is

\[ A_{pt, i}^{HTI} = C_{b, i} \cos \Delta \varphi_{o,i} t_{o,i} V_{ver,i}^2. \]  

(E9)

For a VTI layer, the offset coefficients are

\[ A_{px, i}^{VTI} = (1 + 2 \delta_2) t_{o,i} V_{ver,i}^2, \quad A_{py, i}^{VTI} = 0, \]  

(E10)

and the traveltime coefficient is

\[ A_{pt, i}^{VTI} = \frac{1 + 2 \delta_2}{2} t_{o,i} V_{ver,i}^2, \]  

(E11)

An isotropic layer can be considered a particular case of a VTI layer, with vanishing Thomsen parameter \( \delta \). Equations (E1)–(E10) lead to the following resulting relationship for the normal moveout (NMO) velocity of the entire package,

\[ t_0 V_{nmo}^2 = (\sum A_{s,i})^2 + (\sum A_{o,i})^2. \]  

(E12)

where for an HTI layer

\[ A_{s,i} = [1 + 2 \delta_{2, i} \cos^2 (\varphi_{ax,i} - \varphi_{ph})] t_{o,i} V_{ver,i}^2, \]  

(E13)

while for a VTI layer

\[ A_{s,i} = \delta_{2, i} \sin 2 (\varphi_{ax,i} - \varphi_{ph}) t_{o,i} V_{ver,i}^2, \]  

(E14)

The azimuth of the surface offset is given by

\[ \tan \varphi_{off} = \sum A_{s,i} / \sum A_{s,i}. \]  

(E15)
Each HTI layer is characterized by its vertical velocity $V_{\text{ver},i}$, ‘orthorhombic’ Thomsen parameter $\delta_{2,i}$ and the azimuth of the axis of symmetry $\varphi_{ax,i}$. Each VTI layer has the vertical velocity and TI Thomsen parameter $\delta$ and each isotropic layer has the vertical velocity only. In addition, each layer of any type has thickness $t_{o,i}$ measured in units of time (one-way time in this study). Our goal is to find the parameters of the effective model (consisting of a unique HTI layer), $V_{\text{ver}}^\text{eff}$, $\delta_{2}^\text{eff}$ and $\varphi_{ax}^\text{eff}$ such that for any azimuth of the phase velocity $\varphi_{\text{phs}}$, the magnitude of the NMO velocity of the package will match the magnitude of the NMO velocity of the effective model. The vertical time of the effective model is just the sum of vertical times of all layers, so equation (E11) becomes

$$\frac{(\sum A_{x,i})^2 + (\sum A_{y,i})^2}{\sum A_{x,i}} = \frac{A_{\text{eff}}^2 + A_{\text{eff}}^2}{A_{\text{eff}}^2}. \quad (E15)$$

For this, we require

$$\sum A_{x,i} = A_{x}^\text{eff}, \quad \sum A_{y,i} = A_{y}^\text{eff}. \quad (E16)$$

Note that if equation (E16) is satisfied, then not only equation (E11) holds but also equation (E14). In other words, this means that the effective model matches not only the magnitude of the normal moveout velocity but also its (lateral) direction. That is to say, the azimuthal deviation $\varphi_{\text{eff}} - \varphi_{\text{phs}}$ will be the same for the original and the effective models.

We first consider that the package consists of HTI layers only (with arbitrary and different azimuthal orientations of their axes of symmetry) and then we will generalize our findings for a package that includes VTI and isotropic layers as well. It follows from equations (E12) and (E16),

$$\sum_{i} [1 + 2\delta_{2,i} \cos 2(\varphi_{ax,i} - \varphi_{\text{phs}})] t_{o,i} V_{\text{ver},i}^2 = [1 + 2\delta_{2}^\text{eff} \cos 2(\varphi_{ax}^\text{eff} - \varphi_{\text{phs}})] t_{o} V_{\text{ver}}^\text{eff2}, \quad (E17)$$

$$\sum_{i} \delta_{2,i} \sin 2(\varphi_{ax,i} - \varphi_{\text{phs}}) t_{o,i} V_{\text{ver},i}^2 = \delta_{2}^\text{eff} \sin 2(\varphi_{ax}^\text{eff} - \varphi_{\text{phs}}) t_{o} V_{\text{ver}}^\text{eff2}. \quad (E17)$$

Consider the first equation of set (E17) and expand the cosines squared,

$$\sum_{i} [1 + \delta_{2,i} + \delta_{2,i} \cos 2(\varphi_{ax,i} - \varphi_{\text{phs}})] t_{o,i} V_{\text{ver},i}^2 = [1 + \delta_{2}^\text{eff} + \delta_{2}^\text{eff} \cos 2(\varphi_{ax}^\text{eff} - \varphi_{\text{phs}})] t_{o} V_{\text{ver}}^\text{eff2}. \quad (E18)$$

Continuing the expansion, we obtain

$$\sum_{i} \delta_{2,i} \sin 2(\varphi_{ax,i} - \varphi_{\text{phs}}) t_{o,i} V_{\text{ver},i}^2 = \delta_{2}^\text{eff} \sin 2(\varphi_{ax}^\text{eff} - \varphi_{\text{phs}}) t_{o} V_{\text{ver}}^\text{eff2}. \quad (E19)$$

We emphasize that equation (E19) should be satisfied for any value of the phase velocity azimuth $\varphi_{\text{phs}}$. Terms with $\cos 2\varphi_{\text{phs}}$, those with $\sin 2\varphi_{\text{phs}}$ and the free terms (which do not include $\varphi_{\text{phs}}$) are linearly independent and they should be balanced apart. Therefore, we balance the corresponding coefficients and obtain three distinct equations,

$$\sum_{i} (1 + \delta_{2,i}) t_{o,i} V_{\text{ver},i}^2 = (1 + \delta_{2}^\text{eff}) t_{o} V_{\text{ver}}^\text{eff2}. \quad (E20)$$

For the VTI and isotropic layers, one can assume that the ‘axis azimuth’ coincides with the azimuth of the phase velocity, $\varphi_{ax,i} = \varphi_{\text{phs}}$. These layers have no effect on the first two equations of set (E20) but contribute additional terms to the third equation of this set,

$$\sum_{i} (1 + \delta_{2,i}) t_{o,i} V_{\text{ver},i}^2 + \sum_{i} (1 + 2\delta_{i}) t_{o,i} V_{\text{ver},i}^2 + \sum_{i} t_{o,i} V_{\text{ver},i}^2 = (1 + \delta_{2}^\text{eff}) t_{o} V_{\text{ver}}^\text{eff2}. \quad (E21)$$

Next we consider the second equation of set (E17) and expand the sines,

$$\sum_{i} \delta_{2,i} (\sin 2\varphi_{ax,i} - \sin 2\varphi_{\text{phs}}) t_{o,i} V_{\text{ver},i}^2 = \delta_{2}^\text{eff} (\sin 2\varphi_{ax}^\text{eff} - \sin 2\varphi_{\text{phs}}) t_{o} V_{\text{ver}}^\text{eff2}. \quad (E22)$$

Again, we balance the linearly independent terms but now this operation does not lead to any new equations. We obtain the first two equations of set (E20), i.e., there is no new info and no contradiction with the existing info. Thus, we need to solve a system of three equations that includes the first two
equations of set (E20) and equation (E21),
\[
\begin{align*}
\delta_2^\text{eff} \cos 2\varphi_{ax}^\text{eff} V_{ver}^2 &= \frac{W_x}{t_{o}^{\text{eff}}}, \\
\delta_2^\text{eff} \sin 2\varphi_{ax}^\text{eff} V_{ver}^2 &= \frac{W_y}{t_{o}^{\text{eff}}},
\end{align*}
\]
(E23)

where
\[
\begin{align*}
W_x &= \sum_{i} \delta_{2,i} \cos 2\varphi_{ax,i} t_{o,i} V_{ver,i}^2, \\
W_y &= \sum_{i} \delta_{2,i} \sin 2\varphi_{ax,i} t_{o,i} V_{ver,i}^2,
\end{align*}
\]
(E24)

and
\[
U = \sum_{i} (1 + 2\delta_{2,i}) t_{o,i} V_{ver,i}^2 + \sum_{i} (1 + 2\delta_{i}) t_{o,i} V_{ver,i}^2 + \sum_{i} t_{o,i} V_{ver,i}^2.
\]
(E25)

Note that for a given layered model parameters, values \(W_x, W_y\) and \(U\) are considered known, as they can be computed using explicit formulae. There may be two solutions of equation set (E23): one for positive effective ‘orthorhombic’ Thomsen parameter \(\delta_2^\text{eff}\) and another one for negative. The two solutions are absolutely equivalent. Although they lead to different parameters of the effective layer, they nevertheless result in the same function ‘normal moveout velocity versus phase velocity azimuth’, \(V_{\text{nmo}}(\varphi_{\text{pha}})\). We consider it is natural to assume that \(\delta_2^\text{eff}\) is negative, because normally this parameter is often negative in HTI layers. However, positive \(\delta_2^\text{eff}\) is not a must, even in the case when all \(\delta_{2,i}\) of the layers are negative. We can choose the sign of \(\delta_2^\text{eff}\) arbitrarily but we need to make an explicit choice. For a negative effective Thomsen delta, the first two equations of set (E23) yield the azimuth of the ‘effective axis of symmetry’,
\[
2\varphi_{ax}^\text{eff} = \arctan \left(- \frac{W_x}{W_y}, -\frac{W_y}{W_x}\right), \quad -\pi/2 < \varphi_{ax}^\text{eff} < \pi/2.
\]
(E26)

where the inverse tangent of two arguments is used, or equivalently,
\[
\cos 2\varphi_{ax}^\text{eff} = -\frac{W_x}{W}, \quad \sin 2\varphi_{ax}^\text{eff} = -\frac{W_y}{W}.
\]
(E27)

where
\[
W \equiv \sqrt{W_x^2 + W_y^2}.
\]
(E28)

Now the effective azimuth can be eliminated from equation set (E23) and the remaining equations are,
\[
\begin{align*}
\delta_2^\text{eff} V_{ver}^2 &= \frac{W}{t_{o}^{\text{eff}}}, \\
V_{ver}^2 &= \frac{U}{t_{o}^{\text{eff}}}.
\end{align*}
\]
(E29)

Finally, the effective delta is
\[
\delta_2^\text{eff} = -\frac{W}{W + U}.
\]
(E30)

Comment: we assume that at least one HTI layer is present in the package, otherwise the effective azimuth of symmetry has no meaning. Moreover, for a set consisting of VTI and isotropic layers only, one can calculate only the resulting NMO velocity (which is also the rms velocity) and one can not separate the effective \(\delta\) from the rms velocity of such package. The NMO velocity of a VTI package is described by a well-known formula (Thomsen 1986),
\[
t_o V_{nimo}^2 = \sum_{i} t_{o,i} V_{ver,i}^2 (1 + 2\delta_{i}).
\]
(E31)

The trade-off between the vertical velocity and parameter \(\delta\) can not be resolved even if the NMO velocities for all three (P, SV and SH) waves from a horizontal reflector are known (Tsvankin and Thomsen 1995). On the other hand, if the vertical velocity is known (e.g., from check shots or well logs), the moveout velocity can be used to obtain \(\delta\) (Alkhalifi and Tsvankin 1995).

In Figs 7 and 8 we plot the NMO velocity for a package of layers. The properties of the layers are given in Table I of Part I, Section ‘Numerical Examples’ (Package 1). The thick line corresponds to the NMO velocity of the package and the thin line in Fig. 7 corresponds to the NMO velocity of the equivalent layer with the effective properties. These graphs were plotted to test the algorithm that establishes the effective model parameters. As expected, the two lines coincide exactly (thus, the thin line in legend of Fig. 7 can not be seen in the graph). The function is periodic with the period \(\pi\) radians.

APPENDIX F. NORMAL MOVEOUT VELOCITY VERSUS RAY AND PHASE VELOCITY AZIMUTH

In this appendix we show that the normal moveout (NMO) velocity function versus phase velocity azimuth can be obtained from the NMO velocity function versus ray velocity azimuth, following simple geometric considerations. The NMO velocity versus ray azimuth for horizontal transversely isotropic medium is given by Tsvankin (1997),
\[
\frac{V_{nimo}^2}{V_{ver}^2} = \frac{1 + 2\delta_2}{1 + 2\delta_2 \sin^2 (\varphi_{ray} - \varphi_{ax})}.
\]
(F1)

Equation (F1) can be rewritten in a different (but still equivalent) form,
\[
\frac{\cos^2 (\varphi_{ray} - \varphi_{ax})}{V_{ver}^2 (1 + 2\delta_2)} + \frac{\sin^2 (\varphi_{ray} - \varphi_{ax})}{V_{ver}^2} = \frac{1}{V_{nimo}^2}.
\]
(F2)
This equation describes an ellipse. Indeed, consider an ellipse with semi-axes $A$ and $B$, as seen in Fig. 9. Its canonical equation is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.$$ \(\text{(F3)}\)

Let $\alpha$ be the azimuth of the elliptic radius $R$, measured from the positive semi-axis $x$, as shown in Fig. 9. Then for any point on the elliptic contour,

$$x = R \cos \alpha, \quad y = R \sin \alpha.$$ \(\text{(F4)}\)

Introducing equation (F4) into (F3), we obtain

$$\frac{\cos^2 \alpha}{A^2} + \frac{\sin^2 \alpha}{B^2} = \frac{1}{R^2}.$$ \(\text{(F5)}\)

Comparing equations (F2) and (F5), we conclude that (F2) describes an elliptic line, where the semi-axes and the radius

Figure 7 NMO velocity diagram for package of layers versus azimuth of phase velocity.

Figure 8 NMO velocity for package of layers versus phase velocity azimuth.
Figure 9 Ellipse of NMO velocity versus ray velocity azimuth.

are

\[ A = V_{ver} \sqrt{1 + 2 \delta^2}, \quad B = V_{ver}, \quad R = V_{nmo}. \] (F6)

Angle \( \alpha \) shows the azimuth of the ray velocity with respect to the azimuth of the symmetry axis. Since the NMO plot shows the wavefront configuration, then the normal to the elliptic line shows the direction of the phase velocity. Let \( \beta \) be the azimuth of the normal. Then

\[ \alpha = \varphi_{ray} - \varphi_{ax}, \quad \beta = \varphi_{phs} - \varphi_{ax}. \] (F7)

We will call \( \alpha \) ‘central angle’ and \( \beta \) ‘normal angle”. Equation (F5) can be resolved for the radius \( R \) (NMO velocity), \( x \) (projection of the NMO velocity on the medium axis of symmetry) and \( y \) (projection of the NMO velocity on the direction normal to the axis of symmetry, i.e., on the isotropic plane),

\[ R = \frac{AB}{\sqrt{A^2 \sin^2 \alpha + B^2 \cos^2 \alpha}}. \] (F8)

\[ x = \frac{AB \cos \alpha}{\sqrt{A^2 \sin^2 \alpha + B^2 \cos^2 \alpha}}, \quad y = \frac{AB \sin \alpha}{\sqrt{A^2 \sin^2 \alpha + B^2 \cos^2 \alpha}}. \] (F9)

The arc length element is

\[ ds = \sqrt{dx^2 + dy^2}. \] (F10)

The arc length derivative reads,

\[ \frac{ds}{d\alpha} = \sqrt{(dx/d\alpha)^2 + (dy/d\alpha)^2}. \] (F11)

According to equation (F9), the derivatives of \( x \) and \( y \) components with respect to the central angle are,

\[ \frac{dx}{d\alpha} = -\frac{A^3 B \sin \alpha}{(A^2 \sin^2 \alpha + B^2 \cos^2 \alpha)^{3/2}}. \] \quad (F12)

\[ \frac{dy}{d\alpha} = \frac{AB^3 \cos \alpha}{(A^2 \sin^2 \alpha + B^2 \cos^2 \alpha)^{3/2}}. \] (F13)

Introducing equation (F12) into (F11), we obtain the arc length derivative

\[ \frac{ds}{d\alpha} = \frac{AB \sqrt{A^4 \sin^2 \alpha + B^4 \cos^2 \alpha}}{(A^2 \sin^2 \alpha + B^2 \cos^2 \alpha)^{3/2}}. \] (F14)

Derivatives \( dx/ds \) and \( dy/ds \) define the direction of the line tangent to the (elliptic) contour. They also define the direction of the normal line, see Fig. 10,

\[ \sin \beta = -\frac{dx}{ds} = -\frac{dx/d\alpha}{ds/d\alpha}; \quad \cos \beta = +\frac{dy}{ds} = +\frac{dy/d\alpha}{ds/d\alpha}. \] (F15)

Introduce equations (F12) and (F13) into (F14),

\[ \cos \beta = \frac{B^2 \cos \alpha}{\sqrt{A^4 \sin^2 \alpha + B^4 \cos^2 \alpha}}. \] (F16)

\[ \sin \beta = \frac{A^2 \sin \alpha}{\sqrt{A^4 \sin^2 \alpha + B^4 \cos^2 \alpha}}. \] (F17)

Equation (F15) can be resolved for the central angle \( \alpha \),

\[ \cos \alpha = \frac{A^2 \cos \beta}{\sqrt{A^4 \cos^2 \beta + B^4 \sin^2 \beta}}, \] (F18)

\[ \sin \alpha = \frac{B^2 \sin \beta}{\sqrt{A^4 \cos^2 \beta + B^4 \sin^2 \beta}}. \] (F19)
Next, we rewrite equation (F1) as

\[ \frac{V_{nmo}^2}{V_{ver}^2} = \frac{1 + 2\delta_2}{1 + 2\delta_2 \sin^2 \alpha}, \]  

(F17)

and substitute equation (F16) into (F17). This leads to

\[ \frac{V_{nmo}^2}{V_{ver}^2} = \frac{1 + 4\delta_2 (1 + \delta_2) \cos^2 \beta}{1 + 2\delta_2 \cos^2 \beta}, \]  

(F18)

and we rewrite equation (F18) as

\[ \frac{V_{nmo}^2}{V_{ver}^2} = \frac{1 + 4\delta_2 (1 + \delta_2) \cos^2 (\phi_{phs} - \phi_{ax})}{1 + 2\delta_2 \cos^2 (\phi_{phs} - \phi_{ax})}. \]  

(F19)

This relationship coincides with equation (10) of Part I for the NMO velocity. Eventually, we can establish the azimuthal shift between the moveout direction and the phase velocity azimuth. We can not obtain the exact relationship studying the elliptic front but we can get the hyperbolic approximation for this shift. Recall that

\[ \sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta, \]  

(F20)

\[ \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta. \]

Introduce equation (F15) into (F20),

\[ \sin (\alpha - \beta) = -\frac{\delta_2 \sin 2\beta}{\sqrt{1 + 4\delta_2 (1 + \delta_2) \cos^2 \beta}}, \]  

(F21)

\[ \cos (\alpha - \beta) = \frac{1 + 2\delta_2 \cos^2 \beta}{\sqrt{1 + 4\delta_2 (1 + \delta_2) \cos^2 \beta}}. \]

Note that

\[ \phi_{ax} - \phi_{phs} = -\beta. \]  

(F22)

and therefore, relationship (F21) comes to

\[ \sin (\phi_{ray} - \phi_{phs}) = \frac{\delta_2 \sin 2(\phi_{ax} - \phi_{phs})}{\sqrt{1 + 4\delta_2 (1 + \delta_2) \cos^2 (\phi_{ax} - \phi_{phs})}}, \]  

(F23)

\[ \cos (\phi_{ray} - \phi_{phs}) = \frac{1 + 2\delta_2 \cos^2 (\phi_{ax} - \phi_{phs})}{\sqrt{1 + 4\delta_2 (1 + \delta_2) \cos^2 (\phi_{ax} - \phi_{phs})}}. \]

This coincides with our previous results in equation (10) of Part I.